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MASSLESS WIGNER PARTICLES IN CONFORMAL FIELD THEORY ARE FREE

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Abstract

We show that the massless particle spectrum in a four-dimensional conformal Haag–Kastler net is generated by a free field subnet. If the massless particle spectrum is scalar, then the free field subnet decouples as a tensor product component.

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1. Introduction

Conformal field theories have been extensively studied in two-dimensional spacetime. There are many examples; certain exact computations are available, and they provide also interesting mathematical structures. On the other hand, from a mathematical point of view, no nonperturbative construction of a single interacting quantum field theory in four-dimensional spacetime is available today. In this paper, instead of constructing models, we try to understand general restrictions on models with a large spacetime symmetry. We prove that, if a conformal field theory in four spacetime dimensions in the operator-algebraic approach (Haag–Kastler net) contains massless particles, then there is a free subnet generating the massless particles. Furthermore, if the massless particles are scalar, then they decouple as a tensor product component. Therefore, massless particles in conformal field theory cannot interact.

Actually, Buchholz and Fredenhagen already proved more than 30 years ago that the S-matrix of a dilation-invariant theory is trivial [12]. Based on this result, Baumann [3] has shown that any dilation-invariant scalar field (in the sense of Wightman) where a complete particle interpretation is available (asymptotic completeness with respect to massless particles) is the Wick product of the free field. Compared to these, our results are not necessarily stronger because we assume conformal invariance. On the other hand, there are more general aspects: our framework is Haag–Kastler nets, and we do not assume either the existence of Wightman fields or asymptotic completeness. In two-dimensional spacetime, triviality of the S-matrix does not necessarily imply that the net is free (second quantized). Indeed, in our previous work [36], we have seen that a two-dimensional conformal net is asymptotically complete with respect to massless waves if and only if it is the tensor product of its chiral components. Hence one may consider the tensor product subnet as the 'particle-like' (or 'wave-like') part. However, chiral components can be highly nontrivial (different from the second quantized net, the U(1)-current net). In comparison, in four dimensions, we prove that the particle spectrum is generated by the free, second quantized net. In particular, if the particles are scalar, the free field subnet which we construct cannot have any nontrivial extension, and hence it must decouple in the full net. This is the operator-algebraic version of the argument given in [2, Section 1]. Relaxing the assumption of asymptotic completeness (with respect to massless particles) is important, because while there are many physical arguments that dilation invariance should imply conformal invariance [16, 28], conformal field theory may contain a massive spectrum (the meaning of 'massive' will be clarified in Section 2.1.4), as one would expect from the maximally supersymmetric Yang–Mills theory, which should be conformal [26].

We stress that our approach is nonperturbative. We make an assumption that there is a nonperturbatively given model as a conformal Haag–Kastler net. The existence of massless particles à la Wigner is defined in the sense that the representation of the spacetime translations has nontrivial spectral projection on the surface of the positive lightcone. In this case, Buchholz has established the existence of asymptotic fields [10]. Besides, operator-algebraic scattering theory has been successfully applied to many massive models in low dimensions. The theory was able to reconstruct the factorizing S-matrix as an invariant of the net [23, 37].

There are more claims that conformal fields with massless particles are free with different assumptions [38, 39]. An advantage of our approach is to avoid any field-theoretic calculation. One of the main tools is the Tomita–Takesaki modular theory applied to conformal nets [7]: Brunetti, Guido, and Longo have shown that the modular group of a double cone is certain conformal transformations which preserve the double cone. This renders the central idea of our arguments geometric, combined with the construction of asymptotic fields by Buchholz [10].

Let us recall a technical conjecture in [10]. In order to obtain asymptotic fields, one had to choose local operators with a certain regularity condition in the momentum space, although Buchholz conjectured that this construction should extend to any local operator. In our application, this restriction is a problem because the regularity condition is not stable under conformal transformations. We remove this restriction and show that the asymptotic fields are covariant under the conformal transformation of the given net.

This paper is organized as follows. In Section 2, we summarize the foundations of conformal nets and the massless scattering theory. The technical conjecture above is proved there. We first state and prove our results on the existence of a free subnet for globally conformal nets in Section 3. This additional assumption greatly reduces the problem and emphasizes the geometric nature of our proof. Section 4 treats the general case, not necessarily globally conformal but conformal. We also prove the decoupling of the free scalar subnet. Finally, we discuss open problems and future directions in Section 5.

2. Preliminaries

2.1. Conformal field theory. A model of quantum field theory is realized as a net of von Neumann algebras. A conformal field theory is a net with the conformal symmetry. We collect here the definitions and results necessary for our analysis.

2.1.1. The conformal group and the extended Minkowski space. We consider $\mathbb{R}^4$, the Minkowski space. A conformal symmetry is a transformation of $\mathbb{R}^4$ which preserves the Lorentz metric $a \cdot b = a_0 b_0 - \sum a_k b_k$ up to a function. Actually, we allow a symmetry to take a meager set out of $\mathbb{R}^4$. Hence we need to consider local actions, following the work by Brunetti, Guido, and Longo [7].

Let G be a Lie group, and let M be a manifold. We say that G **acts locally on** M if there is an open nonempty set $B \subset G \times M$ and there is a smooth map $T : B \to M$ such that the following hold.

(1) For any $a \in M$, $V_a := \{g \in G : (g, a) \in B\}$ is an open connected neighborhood of the unit element e of G.

(2) $T_e a = a$ for any $a \in M$.

(3) For $(g, a) \in B$, it holds that $V_{T_g a} = V_a g^{-1}$ and, for $h \in G$ such that $hg \in V_a$, one has $T_h T_g a = T_{hg} a$.

In the following, we only consider $M = \mathbb{R}^4$. The **conformal group** $\mathscr{C}$ is generated by the Poincaré group, dilation, and the special conformal transformations: a special conformal transformation is of the form $\rho\tau(a)\rho$, where $\tau(a)$ is a translation by $a \in \mathbb{R}^4$ and ρ is the relativistic ray inversion

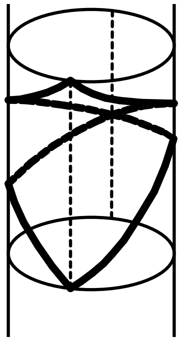

Figure 1. The global space $\widetilde{M}$ projected on the two-dimensional cylinder. The region surrounded by thick lines is a copy of the Minkowski space.

$$\rho a = -\frac{a}{a \cdot a}.$$

This action is quasi global in the sense that for any $g \in \mathscr{C}$ the open set $\{a \in M : (g, a) \in B\}$ is the complement of a meager set S_g and it holds for $a_0 \in S_g$ that $\lim_{a \to a_0} T_g a = \infty$. In other words, the set of points in M which are taken out of M by g is meager. This action T is transitive. It has been shown [7, Propositions 1.1, 1.2] that there is a manifold $\bar{M}$ such that M is a dense open subset of $\bar{M}$ and the action T extends to a transitive global action on $\bar{M}$. Furthermore, the action of T lifts to a transitive global action $\widetilde{T}$ of the universal covering group $\widetilde{G}$ of G on the universal covering $\widetilde{M}$ of $\bar{M}$.

We can realize $\bar{M}$ concretely in $\mathbb{R}^6$ as follows:

$$N := \{(\xi_0, \ldots, \xi_5) \in \mathbb{R}^6 \setminus \{0\} : \xi_0^2 - \xi_1^2 - \cdots - \xi_4^2 + \xi_5^2 = 0\}/\mathbb{R}^*,$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ acts on $\mathbb{R}^6$ by multiplication. For $a \in M = \mathbb{R}^4$, we define the embedding by $\xi_k = a_k$ for $k = 0, 1, 2, 3$ and $\xi_4 = (1 - a \cdot a)/2$, $\xi_5 = (1 + a \cdot a)/2$. The group $\mathrm{PSO}(4, 2)$ acts on N, and this corresponds to the action of the conformal group $\mathscr{C}$. Since the image of M in N is dense, it follows that $N = \bar{M}$ [7]. One observes that N is diffeomorphic to $(S^3 \times S^1)/\mathbb{Z}_2$; hence its universal covering is $S^3 \times \mathbb{R}$ (see Figure 1).

2.1.2. Conformal nets. An operator-algebraic conformal field theory, or a **conformal net**, is a triple $(\mathcal{A}, U, \Omega)$ of a map $\mathcal{A}$ from the family of open double cones in M into the family of von Neumann algebras on $\mathcal{H}$, a local unitary representation (the group structure is respected only locally) U of the conformal group $\mathscr{C}$, and a unit vector $\Omega \in \mathcal{H}$ such that the following hold.

(1) **Isotony.** If $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

(2) **Locality.** If O_1 and O_2 are spacelike separated, then $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute.

(3) **Local conformal covariance.** For each double cone $O \subset M$, there is a neighborhood V_O of the identity of $\mathscr{C}$ such that $V_O \times O \subset B$, where B is the domain of the local action of $\mathscr{C}$ on M, such that $\mathrm{Ad}\, U(g)(\mathcal{A}(O)) = \mathcal{A}(gO)$.

(4) **Positivity of energy.** The spectrum of the subgroup of translations in $\mathscr{C}$ in the representation U (this is well defined although the action U is local, since the group of translations is simply connected) is included in the closed positive lightcone $\overline{V}_+ := \{a \in \mathbb{R}^4 : a_0 \geqslant 0,\ a \cdot a \geqslant 0\}$.

(5) **Vacuum.** The vector Ω is invariant under the action of U. Such a vector is unique up to a scalar.

(6) **Reeh–Schlieder property.** The vector Ω is cyclic and separating for each local algebra $\mathcal{A}(O)$.

Note that the Reeh–Schlieder property is usually proved under additivity. We take it here as an assumption for simplicity (see the discussion in [40, Section 2]).

A conformal net can be extended to $\widetilde{M}$ with the action of $\widetilde{\mathscr{C}}$ [7, Proposition 1.9]. Indeed, the representation U lifts to $\widetilde{\mathscr{C}}$, and the local algebra $\mathcal{A}(O)$ for O which is not included in $\widetilde{M}$ is defined by covariance.

A **(conformal) subnet** $\mathcal{A}_0$ of a net $(\mathcal{A}, U, \Omega)$ is a family of von Neumann subalgebras $\mathcal{A}_0(O) \subset \mathcal{A}(O)$ such that isotony and covariance with respect to the same U hold. In this case, $\overline{\mathcal{A}_0(O)\Omega}$ is a Hilbert subspace of $\mathcal{H}$ independent of O.

2.1.3. Bisognano–Wichmann property. Certain regions play a special role in the study of conformal field theory. Here, we pick the standard wedge in the a_1-direction, the unit double cone, and the future lightcone:

- $W_1 := \{a \in M : a_1 > |a_0|\}$,
- $O_1 := \left\{a \in M : |a_0| + \sqrt{a_1^2 + a_2^2 + a_3^2} < 1\right\}$,
- $V_+ := \{a \in M : a_0 > 0,\ a \cdot a > 0\}$.

To each of these regions O in $\widetilde{M}$ we associate a one-parameter group Λ_t^O in $\widetilde{\mathscr{C}}$ which preserves O and commutes with all O-preserving conformal transformations.

- For the wedge W_1, we take the boosts in the a_1-direction. They are linear transformations, and their actions on (a_0, a_1) components can be written, in a matrix form, as $\Lambda_t^{W_1} = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t \\ -\sinh 2\pi t & \cosh 2\pi t \end{pmatrix}$.
- For the unit double cone, by rotation invariance the action is determined by the action on the (a_0, a_1)-plane:

$$\Lambda_t^{O_1} a_\pm = \frac{(1 + a_\pm) - e^{-2\pi t}(1 - a_\pm)}{(1 + a_\pm) - e^{-2\pi t}(1 + a_\pm)},$$

  where $a_\pm = a_0 \pm a_1$.
- For the future lightcone V_+, we take the dilation: $\Lambda_t^{V_+} a = e^{2\pi t} \cdot a$.

These regions are mapped to each other by conformal transformations (on $\widetilde{M}$), and the associated transformations are coherent, in the sense that $\Lambda_t^{O} = g^{-1} \Lambda_t^{O'} g$, where $O = gO'$, $g \in \widetilde{\mathscr{C}}$, and $O, O' = W_1, O_1, V_+$. One can define Λ_t^O for any other double cone, wedge, or lightcone by coherence.

For a conformal net, the modular group of a local algebra with respect to the vacuum has been completely determined [7].

THEOREM 2.1 (Bisognano–Wichmann property). *Let $(\mathcal{A}, U, \Omega)$ be a conformal net, and consider its natural extension to $\widetilde{M}$. Then, for any image O of a double cone by a conformal transformation in $\widetilde{\mathscr{C}}$, one has $\Delta_O^{it} = U(\Lambda_t^O)$, where Δ_O is the modular operator of $\mathcal{A}(O)$ with respect to Ω.*

The following duality has been also proved [7].

THEOREM 2.2 (Haag duality on $\widetilde{M}$). *Let $(\mathcal{A}, U, \Omega)$ be a conformal net, and consider its natural extension to $\widetilde{M}$. Then, for a wedge W, it holds that $\mathcal{A}(W)' = \mathcal{A}(W')$.*

Since a conformal transformation can bring a wedge to a double cone O, a similar duality holds for double cones. In that case, we need the causal complement O^{c} on $\widetilde{M}$ rather than the usual spacelike complement O' (see Figure 2).

2.1.4. Representation theory of the conformal group. The conformal group is locally isomorphic to SU(2, 2), and its unitary positive-energy irreducible representations have been classified [25]. Using the dimension $d \geqslant 0$ and half-integers $j_1, j_2 \geqslant 0$, they are parameterized as follows. When restricted to the Poincaré group, one can consider the mass parameter m and spin s or helicity.

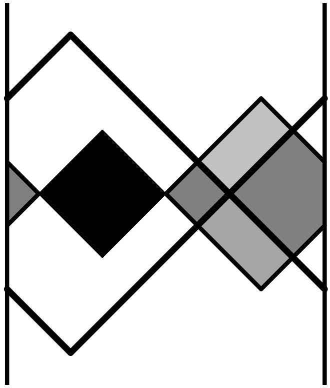

Figure 2. Regions in the global space $\widetilde{M}$. The left and right sides are identified. The white square: a copy of the Minkowski space. Black: a double cone O. Dark gray: the spacelike complement O' of the double cone in the Minkowski space. Light gray + dark gray: the causal complement O^{c} in $\widetilde{M}$.

- Trivial representation. $d = j_1 = j_2 = 0$.
- $j_1 \neq 0 \neq j_2$, $d > j_1 + j_2 + 2$. In this case, $m > 0$ and $s = |j_1 - j_2|, \dots j_1 + j_2$ (integer steps).
- $j_1 j_2 = 0$, $d > j_1 + j_2 + 1$. $m > 0$ and $s = j_1 + j_2$.
- $j_1 \neq 0 \neq j_2$, $d = j_1 + j_2 + 2$. $m > 0$ and $s = j_1 + j_2$.
- $j_1 j_2 = 0$, $d = j_1 + j_2 + 1$. $m = 0$ and helicity $s = j_1 - j_2$.

Hence, the only massless representations are the last family. In this paper, when we say that a conformal net contains massless particles, it means that the representation U has a subrepresentation in this family.

In [39], the following has been proved: if there is a quantum field (an operator-valued distribution) which transforms as a vector in one of the above massless representations, then it is free. This implicitly assumes that the massless particles are generated by such a field. This is apparently a stronger assumption than the one in the operator-algebraic approach (see Section 2.2) that local observables generate states which contain massless particles.

The other nontrivial representations have mass $m > 0$. One can call them massive, although there is no mass gap because of the action of dilations.

2.2. Massless scattering theory. In the operator-algebraic approach, the concept of a particle is not given *a priori*, but has to be defined through operational process. Such a theory for massless particles has been established in [10] for a Poincaré covariant net under the assumption that the representation of the translation has nontrivial spectral projection corresponding to the cone $m = 0$. In such a case, we say that the net contains massless particles (following Wigner).

2.2.1. Convergence of asymptotic fields for regular operators. Let $(\mathcal{A}, U, \Omega)$ be a Poincaré covariant net (a net for which the covariance is only assumed for the Poincaré group). Let x be an operator in $\mathcal{A}(O)$ which is smooth in norm under the group action $g \mapsto \mathrm{Ad}\, U(g)(x)$. There are sufficiently many such operators. Indeed, if x is localized in a slightly smaller region than O, then one can smear x with a smooth function with compact support in the group (note that the conformal group $\mathscr{C}$ is finite dimensional). For a vector $a \in M$, we denote $x(a) = \mathrm{Ad}\, U(\tau(a))(x)$. For $t \in \mathbb{R}$, we define

$$\Phi^t(x) := -2t \int_{S^2} d\omega(\mathbf{n})\ \partial_0 x(t, t\mathbf{n}),$$

where $d\omega$ is the normalized rotation-invariant measure on S^2 and ∂_0 is the derivative with respect to the time translation (which is independent from t). By a straightforward calculation, one finds that

$$\Phi^t(x)\Omega = \frac{1}{|\mathbf{P}|}(e^{it(H-|\mathbf{P}|)} - e^{it(H+|\mathbf{P}|)})Hx\Omega,$$

where $P = (H, \mathbf{P})$ is the generator of translation: $U(\tau(a)) = e^{itP\cdot a}$. Furthermore, we need to take suitable time averages. We fix a positive, smooth, and compactly supported function h with $\int_{\mathbb{R}} h(t)dt = 1$ and $h_T(t) = \frac{1}{\log|T|} h\left(\frac{t-T}{\log|T|}\right)$. We set

$$\Phi^{h_T}(x) = \int_{\mathbb{R}} dt\ h_T(t)\Phi^t(x).$$

Then, by the mean ergodic theorem, one obtains [11]

$$\operatorname*{s-lim}_{T\to\infty} \Phi^{h_T}(x)\Omega = P_1 x\Omega,$$

where P_1 is the projection onto the massless one-particle space, where $H = |\mathbf{P}|$ holds (see Figure 3).

For any double cone O, we denote by $V_{O,+}$ the future tangent of O, the set of all points separated by a future-timelike vector from any point of O. For a fixed double cone O_+ in $V_{O,+}$, there is a sufficiently large T such that $\Phi^{h_T}(x)$ is contained in the causal complement of O_+. In particular, for sufficiently large T, there is a large commutant for $\Phi^{h_T}(x)$, and one can define the operator $\Phi^{\mathrm{out}}(x)$

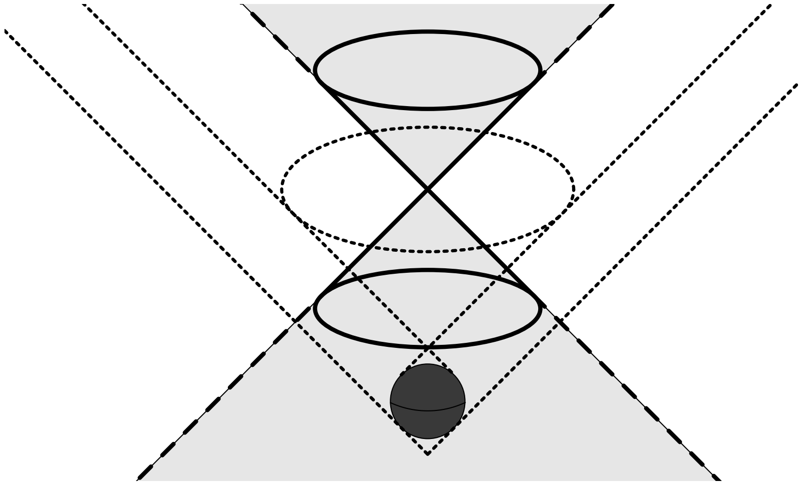

Figure 3. This figure shows how asymptotic fields are constructed. A local observable in a dark gray region is taken in the region between the cones indicated by dotted lines.

by $\Phi^{\mathrm{out}}(x)y\Omega = \text{s-}\lim_{T\to\infty} y\Phi^{h_T}(x)\Omega = yP_1x\Omega$, where $y \in \mathcal{A}(O_+)$. Let us denote $\mathcal{F}(V_{O,+}) = \bigcup_{O_+\subset V_{O,+}} \mathcal{A}(O_+)$ (the union, not the weak closure, and O_+ are bounded). The choice of O_+ was arbitrary in $V_{O,+}$; hence $\Phi^{\mathrm{out}}(x)$ can be defined on $\mathcal{F}(V_{O,+})\Omega$. It is easy to see that $\Phi^{\mathrm{out}}(x)$ is closable. We denote the closure by the same symbol, and its domain by $\mathcal{D}(\Phi^{\mathrm{out}}(x))$. For $N \in \mathbb{N}$, let $\mathcal{A}_N(O)$ be the linear span of the operators

$$\int_{\mathbb{R}} dt\ \varphi(t)\mathrm{Ad}\,U(\tau(ta))(x),$$

where $x \in \mathcal{A}(\check{O})$, a is a timelike vector, and φ is a test function with compact support which has a Fourier transform $\tilde{\varphi}(p)$ with an N-fold zero at $p = 0$, and $\check{O} + (\mathrm{supp}\,\varphi)a \subset O$.

The following has been proved [10, Lemma 1, Lemma 6, Theorems 7, 8, 9].

THEOREM 2.3 (Buchholz). *Let $x = x^*$ be an element of $\mathcal{A}_{N_0}(O)$, where $N_0 \geqslant 15$, O is a double cone, and $V_{O,+}$ be the future tangent of O. Then the following hold.*

(1) *For an arbitrary $y \in \mathcal{A}(O_+)$, where $O_+ \subset V_{O,+}$ is bounded, $y \cdot \mathcal{D}(\Phi^{\mathrm{out}}(x)) \subset \mathcal{D}(\Phi^{\mathrm{out}}(x))$, and one has $[\Phi^{\mathrm{out}}(x), y] = 0$ on $\mathcal{D}(\Phi^{\mathrm{out}}(x))$.*

(2) *The operator* $\Phi^{\mathrm{out}}(x)$ *is self-adjoint and depends only on* $P_1 x\Omega$*. The subspace* $\mathcal{F}(V_{O,+})\Omega$ *is a core of* $\Phi^{\mathrm{out}}(x)$*.*

(3) *The sequence* $\Phi^{h_T}(x)$ *is convergent to* $\Phi^{\mathrm{out}}(x)$ *in the strong resolvent sense.*

(4) *The operator* $\Phi^{\mathrm{out}}(x)$ *can be applied to the vacuum* Ω *arbitrarily many times. We denote the vectors generated in this way recursively (the first term on the right-hand side which contains* $n+1$ *product is defined in this way):*

$$\Phi^{\mathrm{out}}(x)\cdot \xi_1 \overset{\mathrm{out}}{\times} \xi_2 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n = \xi \overset{\mathrm{out}}{\times} \xi_1 \overset{\mathrm{out}}{\times} \xi_2 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n + \sum_{k=1}^{n} \langle \xi, \xi_k\rangle \xi_1 \overset{\mathrm{out}}{\times} \cdots \check{\xi}_k \cdots \overset{\mathrm{out}}{\times} \xi_n,$$

where $\xi = P_1 x\Omega = P_1 x^*\Omega$ *and* $\check{\xi}_k$ *means the omission of the kth element. Then the symbol* $\overset{\mathrm{out}}{\times}$ *is compatible (unitarily equivalent) with the normalized symmetric tensor product on the Fock space with the one-particle space* $P_1\mathcal{H}$*. The domain of* $\Phi^{\mathrm{out}}(x)$ *includes the set* $\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$ *of all linear combinations (without closure) of product states* $\xi_1 \overset{\mathrm{out}}{\times} \xi_2 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$*, where* ξ_k *is an arbitrary vector in* $P_1\mathcal{H}$*.*

(5) *It holds that* $\mathrm{Ad}\, U(g)(\Phi^{\mathrm{out}}(x)) = \Phi^{\mathrm{out}}(\mathrm{Ad}\, U(g)(x))$ *if* g *is a Poincaré transformation.*

(6) *For the resolvent* $R_{\pm i}(y) = (y \pm i)^{-1}$ *of* y*, it holds that*

$$\begin{aligned}
&[R_{\pm i}(\Phi^{\mathrm{out}}(x_1)), R_{\pm i}(\Phi^{\mathrm{out}}(x_2))] \\
&= \langle \Omega, [\Phi^{\mathrm{out}}(x_1), \Phi^{\mathrm{out}}(x_2)]\Omega\rangle \cdot R_{\pm i}(\Phi^{\mathrm{out}}(x_1)) R_{\pm i}(\Phi^{\mathrm{out}}(x_2))^2 R_{\pm i}(\Phi^{\mathrm{out}}(x_1)) \\
&= \mathrm{Re}\,\langle P_1 x\Omega, P_1 x_2\Omega\rangle \cdot R_{\pm i}(\Phi^{\mathrm{out}}(x_1)) R_{\pm i}(\Phi^{\mathrm{out}}(x_2))^2 R_{\pm i}(\Phi^{\mathrm{out}}(x_1)),
\end{aligned}$$

where Re *denotes the real part of the following number.*

(7) *For* $x \in \mathcal{A}_{N_0}(O)$ *and* $y \in \mathcal{F}(V_{O,+})$*, it holds that* $[R_{\pm i}(\Phi^{\mathrm{out}}(x)), y] = 0$.

We note that by Claims (1) and (3), the domain of $\Phi^{\mathrm{out}}(x)$ includes $\mathcal{F}(V_{O,+})\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$.

The restriction to $\mathcal{A}_{N_0}$ is essential in the original proof [10]. The technical issue is that the set $\mathcal{A}_{N_0}(O)$ is covariant under Poincaré transformations and dilations but not under conformal transformations. We will extend these results to each smooth operator in a local algebra $\mathcal{A}(O)$. This has been expected by Buchholz himself in the same paper [10, P. 157, footnote].

2.2.2. *Extension to general smooth operators.* We exploit the arguments of [32, Chapter VIII.7] and [31, Chapter X.10]. Let $\{A_n\}$ be a sequence of (unbounded) operators. The following is an adaptation of [31, Theorem X.63] to the case of our interest.

LEMMA 2.4. *Let $\{A_n\}$ be a sequence of self-adjoint operators on $\mathcal{H}$, whose domains have a dense intersection $\mathcal{D}$, and suppose that their resolvents $R_{\pm i}(A_n)$ are strongly convergent, whose limits we denote by $R_\pm$, and that, for each $\xi \in \mathcal{D}$, $A_n\xi$ is convergent in norm, whose limit we denote by $A\xi$. Then there is a self-adjoint extension $\tilde{A}$ of A, and A_n is convergent to $\tilde{A}$ in the strong resolvent sense.*

Proof. We claim that $\ker R_\pm = \{0\}$. Let $\xi \in \ker R_+$ and let $\eta \in \mathcal{D}$. It is clear that $R_+^* = R_-$. It holds that

$$\begin{aligned}
\langle \xi, \eta \rangle &= \langle \xi, R_{-i}(A_n)(A_n - i)\eta \rangle \\
&= \langle R_{+i}(A_n)\xi, (A_n - i)\eta \rangle \\
&= \lim_n \langle R_{+i}(A_n)\xi, (A_n - i)\eta \rangle \\
&= \langle R_+\xi, (A - i)\eta \rangle \\
&= 0.
\end{aligned}$$

As $\mathcal{D}$ is dense, $\xi = 0$. Similarly, $\ker R_- = \{0\}$, and it follows that $\operatorname{Ran} R_\pm$ are dense in $\mathcal{H}$ since $R_\pm = R_\mp^*$. Then, by the Trotter–Kato theorem [32, Theorem VIII.22], there is a self-adjoint operator $\tilde{A}$, and $A_n \to \tilde{A}$ in the strong resolvent sense.

The domain of $\tilde{A}$ is exactly $R_\pm\mathcal{H}$, and for $\xi \in \mathcal{D}$ it holds that

$$R_\pm \cdot (A \pm i)\xi = \lim_n R_{\pm i}(A_n)(A_n \pm i)\xi = \xi,$$

by the uniform boundedness of $R_{\pm i}(A_n)$; hence ξ is in the range of $R_\pm$, and $\mathcal{D}$ is included in the domain of $\tilde{A}$. □

We do not know whether $\mathcal{D}$ is a core of $\tilde{A}$ in general. We will prove this in the case of asymptotic fields.

Let $N_0 \geqslant 15$. For a smooth $x \in \mathcal{A}(O)$, where O is a double cone, there is a sequence $x_n \in \mathcal{A}_{N_0}(O_n)$ such that $P_1 x\Omega = \lim P_1 x_n \Omega$ and $P_1 x^*\Omega = \lim P_1 x_n^* \Omega$ by the argument of [10, Remark, p.155], where $\{O_n\}$ is growing to the past of O. Namely, for $n \in \mathbb{N}$, one can take $\varphi_n(t)$, whose Fourier transform is

$$\tilde{\varphi}_n(\omega) = (1 + (e^{-i\omega n} - 1)/i\omega n)^{N_0} \cdot \tilde{\varphi}(\omega/n),$$

where φ is a test function which vanishes for $t \geqslant 0$ and $\int dt\, \varphi(t) = 1$. We define $x_n = \int dt\, \varphi_n(t) \mathrm{Ad}\, U(\tau(t,0))(x)$, where τ denotes the translation. If x is self-adjoint, we may consider $x_n + x_n^*$, and assume that x_n are self-adjoint as well. It is clear that x_n are contained in the union of past translations of O. Let O_n be their localization regions. Let $V_{O,+}$ be the future tangent of O; then it is the future tangent of the finite union $O \cup O_1 \cup \cdots \cup O_n$. By [10, Theorem 7] cited above, all $\{\Phi^{\mathrm{out}}(x_n)\}$ are self-adjoint. In addition, $\mathcal{F}(V_{O,+})\Omega$, and accordingly $\mathcal{F}(V_{O,+})\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$, are common cores.

LEMMA 2.5. *The sequence $\{\Phi^{\mathrm{out}}(x_n)\}$ is convergent in the strong resolvent sense.*

Proof. Let us denote $R_{\pm,n} = R_{\pm i}(\Phi^{\mathrm{out}}(x_n))$. On the subspace $\{y\Omega : y \in \mathcal{F}(V_{O,+})\}$, which is a common core for $\{\Phi^{\mathrm{out}}(x_n)\}$, it holds that $R_{\pm,n} y\Omega = yR_{\pm,n}\Omega$ and $y \in \mathcal{F}(V_{O,+})$ is bounded. Since $\{R_{\pm,n}\}$ is uniformly bounded, it is enough to show that $\{R_{\pm,n}\Omega\}$ is convergent.

We know from [10] that $\Phi^{\mathrm{out}}(x_n)$ acts on $\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$ like the free field. Since the problem is now reduced to the vacuum Ω and the free fields, we can restrict ourselves to $\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$ and its closure, namely the Fock space generated from Ω by the fields. Let us denote $\xi_n := P_1 x_n \Omega$. The action of the exponentiated field $e^{i\Phi^{\mathrm{out}}(x_n)}$ on the vacuum Ω is given by $e^{i\Phi^{\mathrm{out}}(x_n)}\Omega = e^{-\frac{1}{2}\langle\xi_n,\xi_n\rangle} e^{\xi_n}$, where we introduced a vector (cf. [24])

$$e^{\eta} := \Omega \bigoplus_k \frac{1}{\sqrt{k!}} \eta^{\otimes k}.$$

It is easy to see that $\langle e^{\eta}, e^{\zeta}\rangle = e^{\langle \eta,\zeta\rangle}$. Now it is obvious that $\eta \mapsto e^{\eta}$ is continuous. This implies the convergence $e^{\xi_n} \to e^{\xi}$ when $\xi_n \to \xi$. The exponentiated field acts by $e^{i\Phi^{\mathrm{out}}(x_n)} e^{\eta} = e^{-\frac{1}{2}\langle\xi_n,\xi_n\rangle} e^{-\langle \xi_n,\eta\rangle} e^{\xi_n+\eta}$, and $\{e^{\eta}\}$ is total in the Fock space. The whole argument applies to $t\xi_n$ for arbitrary $t \in \mathbb{R}$, and hence $\{e^{it\Phi^{\mathrm{out}}(x_n)}\}$ is strongly convergent to $W(t\xi)$ on the Fock space (because this sequence is uniformly bounded), where $W(\xi)$ is an operator which acts by $W(\xi)\eta = e^{-\frac{1}{2}\langle\xi,\xi\rangle} e^{-\langle\xi,\eta\rangle} e^{\xi+\eta}$. Hence we obtain the convergence in the strong resolvent sense [31, Theorem VIII.21]; in particular, $\{R_{\pm,n}\Omega\}$ is convergent. □

As seen from Theorem 2.3(3), $\{\Phi^{\mathrm{out}}(x_n)\}$ is convergent on $\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$, and hence on $\mathcal{F}(V_{O,+})\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$.

By Lemma 2.4, there is a self-adjoint operator, which we denote by $\Upsilon(\xi)$, such that $\Upsilon(\xi)$ is the limit of $\{\Phi^{\mathrm{out}}(x_n)\}$ in the strong resolvent sense. Accordingly, $\Upsilon(\xi)$ commutes with $\mathcal{F}(V_{O,+})$ on its domain. Importantly, we have shown that $\Upsilon(\xi)$ is a self-adjoint extension of the limit of the sequence $\{\Phi^{\mathrm{out}}(x_n)\}$ on a common domain $\mathcal{F}(V_{O,+})\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$. Furthermore, the action of $\Upsilon(\xi)$ is determined by ξ as in Theorem 2.3(3). This implies that Ω is in the domain of $\Upsilon(\xi)^m$ for any $m \in \mathbb{N}$.

LEMMA 2.6. *Any vector $y\Omega \in \mathcal{F}(V_{O,+})\Omega$ is an analytic vector for $\Upsilon(\xi)$. In particular, $\mathcal{F}(V_{O,+})\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$ is a core of $\Upsilon(\xi)$.*

Proof. We have to estimate $\Upsilon(\xi)^k y\Omega$. The operator $\Upsilon(\xi)$ commutes with y and acts on Ω as the free field. Hence we have

$$\|\Upsilon(\xi)^m y\Omega\| \leqslant \|y\| \cdot \left(\sqrt{(2m)!\,2^{-m}(m!)^{-1}}\right) \cdot \|\xi\|^m.$$

Then it is easy to see that $\sum_m \|\Upsilon(\xi)^m y\Omega\| t^m/m!$ is finite for any t, and, since the subspace $\mathcal{F}(V_{O,+})\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$ of the domain is stable under $\Phi^{\mathrm{out}}(\xi)$, by Nelson's analytic vector theorem [31, Theorem X.39, Corollary 2] (the stability of the domain is important; We thank D. Buchholz for pointing out this assumption.), $\mathcal{F}(V_{O,+})\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$ is a core of $\Upsilon(\xi)$. □

LEMMA 2.7. *The subspace $\mathcal{F}(V_{O,+})\Omega$ is a core of $\Upsilon(\xi)$.*

Proof. In [10, Lemma 6], it was shown that, if $x_0 \in \mathcal{A}_{N_0}(O)$, $N_0 \geqslant 15$, then the domain $\mathcal{D}(\Phi^{\mathrm{out}}(x_0))$ of $\Phi^{\mathrm{out}}(x_0)$, which is defined as the closure of the operator on $\mathcal{F}(V_{O,+})\Omega$, includes $\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$, and the action of $\Phi^{\mathrm{out}}(x_0)$ on $\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$ is exactly same as that of the free fields. Actually, the only properties of $\Phi^{\mathrm{out}}(x_0)$ used there are those that Ω is in the domain of $\Phi^{\mathrm{out}}(x_0)^*\Phi^{\mathrm{out}}(x_0)$ and $\Phi^{\mathrm{out}}(x_0)$ commute with $\mathcal{F}(V_{O,+})$, which are true also for $\Upsilon(\xi)$, as we have seen.

For the reader's convenience, we review the proof of [10, Lemma 6]. Let $x_0 \in \mathcal{A}_{N_0}(O)$. There is an N (depending on n which appears later) such that there is a sequence $\{y_k\}$ which belongs to $\mathcal{A}_N(O_k)$, where $O_k \subset V_{O,+}$ (the localization region O_k depends on k), $y_k\Omega \to \xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$ weakly, and $y_k^* y_k\Omega$ is uniformly bounded. To see that $\xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$ is in the domain of $\Phi^{\mathrm{out}}(x_0)$, one needs to estimate $\langle \Phi^{\mathrm{out}}(x_0)^*\eta, y_k\Omega\rangle$ for an arbitrary vector $\eta \in \mathcal{D}(\Phi^{\mathrm{out}}(x_0)^*)$. By using the fact that $\Phi^{\mathrm{out}}(x_0)$ commutes with y_k (which is also valid for $\Upsilon(\xi)$), one obtains

$$\begin{aligned} |\langle \Phi^{\mathrm{out}}(x_0)^*\eta, y_k\Omega\rangle|^2 &\leqslant \|\eta\|^2 \cdot \|\Phi^{\mathrm{out}}(x_0) y_k\Omega\|^2 \\ &\leqslant \|\eta\|^2 \cdot \|y_k^* y_k\Omega\| \cdot \|\Phi^{\mathrm{out}}(x_0)^*\Phi^{\mathrm{out}}(x_0)\Omega\|, \end{aligned}$$

if $\Phi^{\mathrm{out}}(x_0)\Omega$ is in the domain of $\Phi^{\mathrm{out}}(x_0)^*$. (This follows in the original proof from the assumption that $x_0 \in \mathcal{A}_{N_0}(O)$, and this is the only point where $N_0 \geqslant 15$ is required. For $\Upsilon(\xi)$, we already know that that one can repeat its action on Ω arbitrarily many times.) This expression is uniformly bounded by the choice of y_k; hence $\langle \Phi^{\mathrm{out}}(x_0)^*\eta, \xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n \rangle$ is bounded by $\|\eta\|$ times a constant, and $\xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$ belongs to $\mathcal{D}(\Phi^{\mathrm{out}}(x_0))$.

In order to get the explicit action of $\Phi^{\mathrm{out}}(x_0)$ on $\xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$ (see Theorem 2.3), one takes a sequence $\{x^{(m)}\}$, where each member belongs to $\mathcal{A}_N(O^{(m)})$, double cones growing to the past of O as in the construction before Lemma 2.5 (it is not explicitly written in the original proof, but N must be chosen corresponding to $2(n+1)$; see also [10, Lemmas 2, 3]). In this computation, the only point is that $\{Px^{(m)}\Omega\}$ can approximate $Px_0\Omega$, which is true also for ξ.

Although $\{\xi_k\}$ are not completely arbitrary since $\xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$ must be the limit of $y_k\Omega$, they form a total set in the free Fock space. Once one has obtained the action of $\Phi^{\mathrm{out}}(x_0)$ on a dense subspace, an arbitrary n-particle vector can be approximated in the n-particle subspace, and the action of $\Phi^{\mathrm{out}}(x_0)$ is continuous there; hence, by the closedness of $\Phi^{\mathrm{out}}(x_0)$, it follows that any vector in $\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$ is in the domain of $\Phi^{\mathrm{out}}(x_0)$. The same argument is valid for $\Upsilon(\xi)$.

Altogether, the closure of the restriction of $\Upsilon(\xi)$ to $\mathcal{F}(V_{O,+})\Omega$ includes $\mathcal{F}(V_{O,+})\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}$, and hence the full domain of $\Upsilon(\xi)$ by Lemma 2.6. This was what we had to prove. □

As $\Phi^{\mathrm{out}}(x)$ is defined as the closure of the operator $\mathcal{F}(V_{O,+})y\Omega \ni \eta \longmapsto yP_1x\Omega$, we can infer that $\Phi^{\mathrm{out}}(x) = \Upsilon(\xi)$.

THEOREM 2.8. *For any $x = x^* \in \mathcal{A}(O)$ smooth, $\Phi^{\mathrm{out}}(x)$ is self-adjoint with a core $\mathcal{F}(V_{O,+})\Omega$, where $V_{O,+}$ is the future tangent of O. The sequence $\Phi^{h_T}(x)$ is convergent to $\Phi^{\mathrm{out}}(x)$ in the strong resolvent sense.*

Proof. By definition, $\Phi^{\mathrm{out}}(x)$ is the closure of the operator $y\Omega \mapsto yP_1x\Omega$ on $\mathcal{F}(V_{O,+})\Omega$. But since $\Upsilon(\xi)(= \Upsilon(P_1x\Omega))$ is self-adjoint and $\mathcal{F}(V_{O,+})\Omega$ is its core, it follows that $\Upsilon(\xi) = \Phi^{\mathrm{out}}(x)$, as their actions coincide on their cores.

As for the convergence, we follow the proof of [10, Theorem 9]. We know that $\mathcal{F}(V_{O,+})\Omega$ is a core for $\Phi^{\mathrm{out}}(x)$, and that it is self-adjoint. For $y \in \mathcal{F}(V_{O,+})$,

$$\begin{aligned}
&\operatorname*{s\text{-}lim}_{T\to\infty} (\Phi^{h_T}(x) + \lambda)^{-1}(\Phi^{\mathrm{out}}(x) + \lambda)y\Omega \\
&\quad = \operatorname*{s\text{-}lim}_{T\to\infty} (\Phi^{h_T}(x) + \lambda)^{-1}(\Phi^{h_T}(x) + \lambda)y\Omega = y\Omega
\end{aligned}$$

by the uniform boundedness of $(\Phi^{h_T}(x) + \lambda)^{-1}$ for a fixed $\lambda \notin \mathbb{R}$. By the self-adjointness of $\Phi^{\mathrm{out}}(x)$, $\{(\Phi^{\mathrm{out}}(x) + \lambda)y\Omega, y \in \mathcal{F}(V_{O,+})\}$ is dense in $\mathcal{H}$, and we obtain the convergence in the strong resolvent sense, again by the uniform boundedness of the sequence. □

LEMMA 2.9. *Let* $(\mathcal{A}, U, \Omega)$ *be a conformal net. For* $x = x^* \in \mathcal{A}(O)$ *smooth, there is an* O_+ *whose closure is contained in the future tangent* $V_{O,+}$ *of* O *such that* $\mathcal{A}(\mathcal{O}_+)\Omega$ *is a core for* $\Phi^{\mathrm{out}}(x)$.

Proof. We work on the extension of $\mathcal{A}$ on $\widetilde{M}$ and the lift of U to $\widetilde{\mathscr{C}}$.

Recall that $V_{O,+}$ is a translation of the future lightcone. Then there is a region D in $\widetilde{M}$ such that the inclusion $V_{O,+} \subset D$ is conformally equivalent to $O_+ \subset V_+$, where O_+ is a double cone whose past apex is the point of origin. Then the conformal transformations associated to V_+, dilations, shrink O_+. Accordingly, the conformal transformations associated to D shrink $V_{O,+}$ to double cones whose past apex is the apex of $V_{O,+}$ (see Figure 2). In this situation, such a transformation also shrinks O.

Let g be a conformal transformation as in the previous paragraph. Now the operator $\Phi^{\mathrm{out}}(\operatorname{Ad} U(g)(x))$ has a core $\mathcal{F}(V_{O,+})\Omega$, and $\operatorname{Ad} U(g)(\Phi^{\mathrm{out}}(x))$ has a core $U(g)\mathcal{F}(V_{O,+})\Omega = \mathcal{F}(gV_{O,+})\Omega$, where $\mathcal{F}(gV_{O,+})$ is defined analogously as $\mathcal{F}(V_{O,+})$. Their actions coincide on $\mathcal{F}(gV_{O,+})\Omega$; namely, for $y \in \mathcal{F}(gV_{O,+})$, they give $y\Omega \mapsto yU(g)P_1x\Omega = yP_1U(g)x\Omega$ (the conformal group preserves $P_1\mathcal{H}$ from the classification of unitary positive-energy representations; see Section 2.1.4). The operator $\Phi^{\mathrm{out}}(\operatorname{Ad} U(g)(x))$ is a self-adjoint extension of $\operatorname{Ad} U(g)(\Phi^{\mathrm{out}}(x))$ which is also self-adjoint; hence they must coincide.

In the discussion above, the domain of $\Phi^{\mathrm{out}}(\operatorname{Ad} U(g)(x))$ naturally includes $\mathcal{A}(gV_{O,+})\Omega$ (note that $\mathcal{A}(gV_{O,+})$ is a von Neumann algebra). Reversing the argument, for any $x \in \mathcal{A}(O)$ there is a sufficiently large double cone O_+ in $V_{O,+}$, whose past apex is the future apex of O, such that $\mathcal{A}(O_+)\Omega$ is a core of $\Phi^{\mathrm{out}}(x)$.

Until now, in this proof, and in Theorem 2.8, regarding the localization, we have used only the assumption that, x is localized in O, a double cone in the past tangent of $V_{O,+}$. By considering $\operatorname{Ad} U(\tau(-a))(x)$, which is localized in $O - a$ for a future-timelike vector a, and translating everything by a after the argument, we see actually that $\mathcal{A}(O_+ + a)\Omega$ is a core of $\Phi^{\mathrm{out}}(x)$. In other words, if x is localized in a double cone, then there is another double cone in the future tangent, separated by a nontrivial timelike vector, whose local operators can generate a core for $\Phi^{\mathrm{out}}(x)$. □

COROLLARY 2.10. *Let* $(\mathcal{A}, U, \Omega)$ *be a conformal net. For* $x = x^* \in \mathcal{A}(O)$ *smooth and* $g \in \widetilde{\mathscr{C}}$ *sufficiently near to the unit element such that* gO *is still a double cone in the Minkowski space* M, *it holds that* $\operatorname{Ad} U(g)(\Phi^{\mathrm{out}}(x)) = \Phi^{\mathrm{out}}(\operatorname{Ad} U(g)(x))$.

Proof. We may assume that x is localized in $\check{O}$, whose closure is still in O. Let $O_+ + a$ be a double cone in $V_{O,+}$ separated from the future apex of O such that

$\mathcal{A}(O_+ + a)\Omega$ is a core for $\Phi^{\mathrm{out}}(x)$ (Lemma 2.9). If $g \in \widetilde{\mathscr{C}}$ is sufficiently near to the unit, we may assume the following.

- $g\check{O} \subset O$.
- gO and $g(O_+ + a)$ are included in $\mathbb{R}^4$.
- There is a double cone $\widehat{O}_+$ which includes $(O_+ + a) \cup g(O_+ + a)$ such that $\widehat{O}_+$ and $g^{-1}\widehat{O}_+$ are in the future tangent $V_{O,+}$ of O.

The set $\mathcal{A}(\widehat{O}_+)\Omega$ is a core of $\mathrm{Ad}\, U(g)(\Phi^{\mathrm{out}}(x))$ and $\Phi^{\mathrm{out}}(\mathrm{Ad}\, U(g)(x))$. But their actions on Ω coincide and they commute with $\mathcal{A}(\widehat{O}_+)$; hence the operators must coincide. This concludes the desired local covariance of $\Phi^{\mathrm{out}}(x)$ with respect to U. □

We can now define the outgoing free field net by

$$\mathcal{A}^{\mathrm{out}}(O) := \{R_\lambda(\Phi^{\mathrm{out}}(x)) : x = x^* \in \mathcal{A}(O) \text{ smooth}, \lambda \notin \mathbb{R}\}''.$$

By Corollary 2.10, this net $\mathcal{A}^{\mathrm{out}}$ is covariant with respect to the unitary representation U for the original net $\mathcal{A}$. The vacuum Ω is in general not cyclic for $\mathcal{A}^{\mathrm{out}}$.

This free field net can be defined for any given net which contains massless particles. We will show that it is a subnet for a given conformal net, namely $\mathcal{A}^{\mathrm{out}}(O) \subset \mathcal{A}(O)$.

3. A proof under global conformal invariance

In this section, we show that a globally conformal net (defined below) contains the second quantization (free) net if it has nontrivial massless particle spectrum. Of course these two assumptions are very strong. We can actually drop global conformal invariance as we will see in Section 4, but here we present a simpler proof in order to clarify the ideas involved. This result should thus be considered as a simplification in the operator-algebraic formulation of [3] with an additional assumption, namely global conformal invariance (GCI). It is a strong property, under which there are indications that the stress–energy tensor is the same as that of the free field [33].

A conformal net $(\mathcal{A}, U, \Omega)$ is said to be **globally conformal** if the extension to $\bar{M}$ (the compactified Minkowski space; see Section 2.1.1) already admits a global action of $\widetilde{\mathscr{C}}$ (cf. [29, 30], where GCI is defined in terms of Wightman functions). Namely, the action of $\widetilde{\mathscr{C}}$ factors through the action of $\mathscr{C}$. For example, the massless free fields with odd integer helicity are globally conformal, while other free fields are not [19, Corollary 3.12].

In this case, any two operators x, y localized in timelike-separated regions commute. Indeed, any pair of timelike-separated regions can be brought into spacelike-separated regions by an action of $\mathscr{C}$.

The first consequence of GCI is the following.

PROPOSITION 3.1. *For a net $\mathcal{A}$ with GCI, it holds that $\mathcal{A}(V_+) = \mathcal{A}(V_-)'$, where $V_\pm$ are the future and past lightcones.*

Proof. As remarked above, it holds that $\mathcal{A}(V_+) \subset \mathcal{A}(V_-)'$ by GCI. The modular group for $\mathcal{A}(V_-)$ with respect to Ω is the dilation [7] (see Section 2.1.3), and thus the modular group for $\mathcal{A}(V_-)'$ with respect to Ω is again dilation (up to a reparameterization). It is clear that $\mathcal{A}(V_+)$ is invariant under dilation.

Let us recall the following simple variant of Takesaki's theorem [34, Theorem IX.4.2]. Assume that $\mathcal{N} \subset \mathcal{M}$ is an inclusion of von Neumann algebras, that Ω is a cyclic separating vector for $\mathcal{M}$, and that the modular group $\mathrm{Ad}\,\Delta^{it}$ for $\mathcal{M}$ with respect to Ω preserves $\mathcal{N}$. Then there is a conditional expectation $E : \mathcal{M} \to \mathcal{N}$ which preserves the state $\langle \Omega, \cdot\, \Omega \rangle$, and this is implemented by the projection P onto the subspace $\overline{\mathcal{N}\Omega}$: $E(x)\Omega = Px\Omega$. In particular, $E(x) = x$ if and only if $x \in \mathcal{N}$.

In our situation, from Takesaki's theorem it follows that $\mathcal{A}(V_+) = \mathcal{A}(V_-)'$ because Ω is cyclic for both algebras by the Reeh–Schlieder property (cf. [36, Appendix A]). Therefore the projection above is trivial, and the two von Neumann algebras must coincide. □

LEMMA 3.2. *For a net $\mathcal{A}$ with GCI, the outgoing free field net $\mathcal{A}^{\mathrm{out}}$ is a subnet of $\mathcal{A}$.*

Proof. Let $O \subset V_-$, and let $O_+ \subset V_+$. In particular, O_+ is in the future tangent of O. By the construction of asymptotic fields, $\Phi^{h_T}(x)$ is in the spacelike complement of $\mathcal{A}(O_+)$ if $x \in \mathcal{A}(O)$; hence we have $R_\lambda(\Phi^{\mathrm{out}}(x)) \in \mathcal{A}(V_+)'$ by the convergence in the strong resolvent sense, and by Proposition 3.1 this is equal to $\mathcal{A}(V_-)$. This implies that $\mathcal{A}^{\mathrm{out}}(V_-) \subset \mathcal{A}(V_-)$.

By conformal covariance with respect to the same representation U (see the end of Section 2.2.2), with the conformal group $\mathscr{C}$ which takes V_- to any double cone O, we obtain $\mathcal{A}^{\mathrm{out}}(O) \subset \mathcal{A}(O)$. □

We summarize the result.

THEOREM 3.3. *Let $(\mathcal{A}, U, \Omega)$ be a globally conformal net, and assume that the massless particle spectrum of U is nontrivial. Then there is a subnet $\mathcal{A}^{\mathrm{out}}$ of $\mathcal{A}$,* *which is isomorphic to the free field net associated to the massless representation. The free subnet $\mathcal{A}^{\mathrm{out}}$ generates the whole massless particle spectrum of U.*

Proof. Almost all statements have been proved above. The whole massless particle spectrum of U is generated by $\mathcal{A}^{\mathrm{out}}$ since $\{P_1 x\Omega : x \in \mathcal{A}(O)\}$ is dense in $P_1\mathcal{H}$ by the Reeh–Schlieder property of $\mathcal{A}$, and we only have to consider the asymptotic fields for self-adjoint elements $x_+ = (x+x^*)/2$ and $x_- = (x-x^*)/2i$. The exponentiated fields $e^{i\Phi^{\mathrm{out}}(x_\pm)}$ are localized in $\mathcal{A}^{\mathrm{out}}(O)$, and the one-particle vectors are obtained by $\frac{\mathrm{d}}{\mathrm{d}t}e^{it\Phi^{\mathrm{out}}(x_\pm)}\Omega$. □

One can analogously define $\mathcal{A}^{\mathrm{in}}$ by taking the limit $T \to -\infty$. Now that we know that the net $\mathcal{A}$ includes a free field subnet, it follows that $\mathcal{A}^{\mathrm{out}} = \mathcal{A}^{\mathrm{in}}$, because we can choose local operators x which create one-particle states from the free subnet. For the free field net, the asymptotic field net is of course itself, so we obtain $\mathcal{A}^{\mathrm{out}} = \mathcal{A}^{\mathrm{in}}$. Accordingly, although one can define the S-matrix on the subspace generated by $\mathcal{A}^{\mathrm{out}} = \mathcal{A}^{\mathrm{in}}$, roughly as the difference between $\xi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$ and $\xi_1 \overset{\mathrm{in}}{\times} \cdots \overset{\mathrm{in}}{\times} \xi_n$ (see [12], and [9] for its two-dimensional variant), it is trivial.

4. A general proof

Finally, let us prove the existence of a free subnet under conformal invariance but not necessarily under global conformal invariance. If a net is not globally conformal, it does not necessarily hold that $\mathcal{A}(V_+)' = \mathcal{A}(V_-)$, and our previous argument does not work. Instead, here we use directed asymptotic fields defined below. As already suggested by Buchholz himself [11, Section 4], Theorem 2.3 can be extended for asymptotic fields with a function f which specifies a direction in which a local observable proceeds asymptotically. Such a directed asymptotic field still has a certain local property, and we can construct a subnet.

4.1. Directed asymptotic fields. For a smooth function f on the unit sphere S^2 such that $f(\mathbf{n}) \geqslant 0$ and $\int_{S^2} d\omega(\mathbf{n})\, f(\mathbf{n}) = 1$, we define

$$\Phi_f^t(x) := -2t \int_{S^2} d\omega(\mathbf{n}) f(\mathbf{n}) \partial_0 x(t, t\mathbf{n}), \qquad \Phi_f^{h_T}(x) = \int_{\mathbb{R}} dt\, h_T(t) \Phi_f^t(x),$$

where the notation is as in Section 2.2.1. In [10], the case where $f = 1$ has been worked out, and it has been suggested in [11] that the whole theory works for a general f. As we need certain extended results, let us discuss the proofs and how they should be modified when f is nontrivial.

First, we explain the following claim [11, Equation (4.3)]:

$$\text{s-}\lim_{T\to\infty} \Phi_f^{h_T}(x)\Omega = P_1 f\left(\frac{\mathbf{P}}{|\mathbf{P}|}\right) x\Omega,$$

where $\mathbf{P}$ is the 3-momentum operator of the given representation U of the net (see Section 2.2.1) and $f(\mathbf{P}/|\mathbf{P}|)$ is defined by functional calculus. This follows from the mean ergodic theorem analogously as in [10, Section 2]. Indeed, this time we have

$$\Phi_f^t(x)\Omega = -\frac{it}{2\pi}\int dE_P \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi\ f(\theta,\varphi)e^{it(H-\mathbf{n}\cdot\mathbf{P})}H(x\Omega)_P,$$

where $P = (H, \mathbf{P})$, $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\varphi)$, and the integral is about $\mathbf{n}$ (on the unit sphere) and the joint spectral decomposition with respect to P, and accordingly $(x\Omega)_P$ is the P-component with respect to it. Since the support of P is included in the closed positive lightcone $\overline{V}_+$, the t-dependent phase vanishes $e^{it(H-\mathbf{n}\cdot\mathbf{P})}$ only on the surface of the cone $H = |\mathbf{P}|$. Instead, on this surface the integral with respect to θ, φ gives $\frac{2\pi}{-it|\mathbf{P}|} f\left(\frac{\mathbf{P}}{|\mathbf{P}|}\right) e^{it(H-|\mathbf{P}|)}$ with additional terms which tend to zero when the limit in the mean ergodic theorem is taken. (This can be explicitly demonstrated by considering a function f which is z-rotation symmetric. A general function can be approximated by sums of such functions with different axis of symmetry in the L^1-norm.) Hence we obtain the formula above.

Only in this paragraph, the propositions and sections refer to those in [10]. Now, Lemma 1 can be modified straightforwardly. Lemma 2 is the main technical ingredient and has been proved in the Appendix. Now, among the statements in the Appendix, the only one in which the spherical integral matters is the Lemma, in which commutators of spherically smeared operators are estimated. Here, the only property essentially used in the estimate is locality of operators and the integrand gets bounded by norm. This means that, if one has to smear the integrand with f, it changes the weight of localization. However, as the integrand is bounded by a norm and no other technique is required, one can simply bound f by a constant in order to adapt the proof. By this bound, the estimate gets simply multiplied by a constant depending on f. This does not affect the rest of the arguments at all. Indeed, this Lemma is used later in the Corollary, and indirectly in Proposition II, where the overall constant is unimportant. Finally, Lemma 2 is proved in Section d), and the overall constant in the estimate does not play any role; hence we obtain the modified Lemma 2. In the rest of the paper, the spherical integral appears only through the correspondence from

x to $P_1 f(\mathbf{P}/|\mathbf{P}|)x\Omega$. Accordingly, one can modify all the propositions of the paper.

Thereafter, one can repeat our argument in order to extend the results from $\mathcal{A}_{N_0}(O)$ to $\mathcal{A}(O)$. In summary, we obtain the following.

THEOREM 4.1. *Let $x = x^*, x_1 = x_1^*, x_2 = x_2^*$ be smooth elements (with respect to $\widetilde{\mathscr{C}}$) of $\mathcal{A}(O)$, let O be a double cone, and let f, f_1, f_2 be smooth functions on S^2.*

(1) *For arbitrary $y \in \mathcal{A}(O_+)$, where $O_+ \subset V_{O,+}$ is bounded, $y \cdot \mathcal{D}(\Phi_f^{\mathrm{out}}(x)) \subset \mathcal{D}(\Phi_f^{\mathrm{out}}(x))$, and one has $[\Phi_f^{\mathrm{out}}(x), y] = 0$ on $\mathcal{D}(\Phi_f^{\mathrm{out}}(x))$.*
(2) *The operator $\Phi_f^{\mathrm{out}}(x)$ is self-adjoint and depends only on $P_1 f(\mathbf{P}/|\mathbf{P}|)x\Omega$. The subspace $\mathcal{F}(V_{O,+})\Omega$ is a core of $\Phi_f^{\mathrm{out}}(x)$.*
(3) *The sequence $\Phi_f^{h_T}(x)$ is convergent to $\Phi_f^{\mathrm{out}}(x)$ in the strong resolvent sense.*
(4) *The domain $\mathcal{D}(\Phi_f^{\mathrm{out}}(x))$ includes the set $\mathcal{H}_{\mathrm{prod}}^{\mathrm{out}}$ of all product states $\xi_1 \overset{\mathrm{out}}{\times} \xi_2 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n$, and its action is*

$$\Phi_f^{\mathrm{out}}(x) \cdot \xi_1 \overset{\mathrm{out}}{\times} \xi_2 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n = \xi \overset{\mathrm{out}}{\times} \xi_1 \overset{\mathrm{out}}{\times} \xi_2 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \xi_n + \sum_{k=1}^{n} \langle \xi, \xi_k \rangle \xi_1 \overset{\mathrm{out}}{\times} \cdots \check{\xi}_k \cdots \overset{\mathrm{out}}{\times} \xi_n,$$

where $\xi = P_1 f(\mathbf{P}/|\mathbf{P}|)x\Omega = P_1 f(\mathbf{P}/|\mathbf{P}|)x^*\Omega$.
(5) *For the resolvent $R_{\pm i}(y) = (y \pm i)^{-1}$ of y, it holds that*

$$\begin{aligned}
&[R_{\pm i}(\Phi_{f_1}^{\mathrm{out}}(x_1)), R_{\pm i}(\Phi_{f_2}^{\mathrm{out}}(x_2))] \\
&= \langle \Omega, [\Phi_{f_1}^{\mathrm{out}}(x_1), \Phi_{f_2}^{\mathrm{out}}(x_2)]\Omega \rangle \cdot R_{\pm i}(\Phi_{f_1}^{\mathrm{out}}(x_1)) R_{\pm i}(\Phi_{f_2}^{\mathrm{out}}(x_2))^2 R_{\pm i}(\Phi_{f_1}^{\mathrm{out}}(x_1)) \\
&= \mathrm{Re} \left\langle P_1 f_1 \left(\frac{\mathbf{P}}{|\mathbf{P}|} \right) x_1 \Omega, P_1 f_2 \left(\frac{\mathbf{P}}{|\mathbf{P}|} \right) x_2 \Omega \right\rangle \\
&\quad \cdot R_{\pm i}(\Phi_{f_1}^{\mathrm{out}}(x_1)) R_{\pm i}(\Phi_{f_2}^{\mathrm{out}}(x_2))^2 R_{\pm i}(\Phi_{f_1}^{\mathrm{out}}(x_1)).
\end{aligned}$$

(6) *For $x \in \mathcal{A}(O)$ and $y \in \mathcal{F}(V_{O,+})$, it holds that $[R_{\pm i}(\Phi_f^{\mathrm{out}}(x)), y] = 0$.*

Other propositions in [10, Section 4] can be appropriately modified, but we state here only what we need.

4.2. Conformal free subnet. Let $\mathcal{A}$ be a conformal net with massless particles. We consider the standard double cone O_1. The following is an easy geometric observation (c.f. [11, P.60]).

LEMMA 4.2. *For a double cone O which is sufficiently spacelike separated from O_1, there is a compact set Σ in S^2 such that $\{a + (t, t\mathbf{n}) : a \in O, \mathbf{n} \in \Sigma,$ t sufficiently large$\}$ is spacelike separated from O_1.*

Let us explain what 'sufficiently separated' means. First, we consider for simplicity the point of origin and a spacelike vector v. We may assume that $v = (v_0, 0, 0, v_3)$, where $|v_0| < v_3$. The vectors in question are of the form

$$\{(v_0 + t, t \sin\theta \cos\phi, t \sin\theta \sin\phi, v_3 + t \cos\theta), t \geqslant 0\}.$$

As one can check easily, these are spacelike for sufficiently large t if $\cos\theta > v_0/v_3$. In general, even if O and O_1 are open regions, if the difference $O_1 - O$ is almost in one direction, then the above arguments works.

From this, we see that certain directed asymptotic fields still have certain locality.

LEMMA 4.3. *For $x \in \mathcal{A}(O)$, where $O \perp O_1$ (spacelike separated), and for a smooth function f such that O and the support of f satisfy the situation of Lemma 4.2, $\Phi_f^{\mathrm{out}}(x)$ is affiliated to $\mathcal{A}(O_1)' = \mathcal{A}(O_1^{\mathrm{c}})$.*

Proof. This follows immediately from the localization of approximants $\Phi_f^{h_T}(x)$ and their convergence to $\Phi_f^{\mathrm{out}}(x)$ in the strong resolvent sense. □

We construct a subnet of $\mathcal{A}$ as follows. First, consider the following:

$$\begin{aligned}\mathcal{A}^{\mathrm{dir}}(O_1^{\mathrm{c}}) :=\{&\mathrm{Ad}\, U(g)(R_\lambda(\Phi_f^{\mathrm{out}}(x))) : \mathrm{Im}\,\lambda \neq 0, g \in \widetilde{\mathscr{C}}(O_1),\\ &x \in \mathcal{A}(O), O \perp O_1, f \text{ as Lemma 4.3}\}'',\end{aligned}$$

where $\widetilde{\mathscr{C}}(O_1)$ is the stabilizer group of O_1 in $\widetilde{\mathscr{C}}$. This is clearly a subalgebra of $\mathcal{A}(O_1^{\mathrm{c}}) = \mathcal{A}(O_1)'$. For any other double cone O in the global space $\widetilde{M}$, we can find $g \in \widetilde{\mathscr{C}}$ such that $O = gO_1^{\mathrm{c}}$. With this g, we define $\mathcal{A}^{\mathrm{dir}}(O) = \mathrm{Ad}\, U(g)(\mathcal{A}^{\mathrm{dir}}(O_1^{\mathrm{c}}))$. This is well defined, because in the definition of $\mathcal{A}^{\mathrm{dir}}(O_1^{\mathrm{c}})$ above g runs in the stability group $\widetilde{\mathscr{C}}(O_1)$.

LEMMA 4.4. *The family $\{\mathcal{A}^{\mathrm{dir}}(O)\}$ is a conformal subnet of $\mathcal{A}$, and it generates $\mathcal{H}^{\mathrm{out}}$ from the vacuum Ω.*

Proof. Covariance of $\mathcal{A}^{\mathrm{dir}}$ holds by definition (and well-definedness). $\mathcal{A}^{\mathrm{dir}}(O)$ is a subalgebra of $\mathcal{A}(O)$, and hence locality follows. Positivity of energy and the properties of vacuum are inherited from those of U and Ω.

Note that the closed subspace $\mathcal{H}^{\mathrm{out}} = \overline{\mathcal{H}^{\mathrm{out}}_{\mathrm{prod}}}$ is invariant under $U(g)$. Indeed, we know already that $\mathcal{A}^{\mathrm{out}}$ is a net whose restriction to the Minkowski space M generates the subspace $\mathcal{H}^{\mathrm{out}}$. Any local algebra $\mathcal{A}^{\mathrm{out}}(O)$, where O is a double cone in M, produces a dense subspace of $\mathcal{H}^{\mathrm{out}}$ from Ω, and, if g is in a small neighborhood of the unit element of $\widetilde{\mathscr{C}}$, then $\mathcal{A}^{\mathrm{out}}(gO)$ is again a local algebra in M and it generates another dense subspace of $\mathcal{H}^{\mathrm{out}}$; thus $\mathcal{H}^{\mathrm{out}}$ is invariant under such $U(g)$. A general element g can be reached as a finite product of such elements, and the invariance follows.

For $O \perp O_1$, the fields $\Phi^{\mathrm{out}}_f(x)$, $x \in \mathcal{A}(O)$ can generate $P_1\chi_\Sigma(\mathbf{P}/|\mathbf{P}|)\mathcal{H}$, where Σ is the compact set in Lemma 4.2 and χ_Σ denotes the characteristic function of Σ. One can patch such Σ to see that the whole one-particle space is spanned by $\Phi^{\mathrm{out}}_f(x)$ which are affiliated to $\mathcal{A}^{\mathrm{dir}}(O_1^{\mathrm{c}})$. Since the second quantization structure is the same, $\overline{\mathcal{A}^{\mathrm{dir}}(O_1^{\mathrm{c}})\Omega}$ includes the whole free Fock space $\mathcal{H}^{\mathrm{out}}$. As $\mathcal{H}^{\mathrm{out}}$ is invariant under $U(g)$, by the construction of $\mathcal{A}^{\mathrm{dir}}(O_1^{\mathrm{c}})$, $\mathcal{H}^{\mathrm{out}}$ is the Hilbert subspace generated by $\mathcal{A}^{\mathrm{dir}}(O_1^{\mathrm{c}})$ from Ω. Then the same holds for an arbitrary double cone by the covariance of $\mathcal{A}^{\mathrm{dir}}$ and the invariance of $\mathcal{H}^{\mathrm{out}}$. This is the Reeh–Schlieder property of $\mathcal{A}^{\mathrm{dir}}$ (as a subnet).

Now we consider the isotony of $\mathcal{A}^{\mathrm{dir}}$. The modular group of $\mathcal{A}(O)$ acts geometrically, and $\mathcal{A}^{\mathrm{dir}}(O)$ is invariant under that by construction. By Takesaki's theorem, there is a conditional expectation E^{dir} from $\mathcal{A}(O)$ to $\mathcal{A}^{\mathrm{dir}}(O)$ implemented by the projection P^{out} onto $\mathcal{H}^{\mathrm{out}}$. It is immediate that this defines a coherent family of conditional expectations in the sense that E^{dir} does not depend on O, because it is implemented by the same projection P^{out}. With this, the isotony of $\mathcal{A}^{\mathrm{dir}}$ follows from the isotony of $\mathcal{A}$. □

PROPOSITION 4.5. *Two nets $\mathcal{A}^{\mathrm{dir}}(O)$ and $\mathcal{A}^{\mathrm{out}}(O)$ coincide, the latter being defined in Section 2.2.2.*

Proof. If $x \in \mathcal{A}(O)$ and $y \in \mathcal{A}(O_1)$, where $O \perp O_1$ and f is chosen for the pair O, O_1 as in Lemma 4.2, then $\Phi^{\mathrm{out}}_f(x)$ and $\Phi^{\mathrm{out}}(y)$, or their resolvents, commute by the techniques of Jost–Lehmann–Dyson representation as in [22, Section 4] [10, Theorem 9]. We know that $\mathcal{A}^{\mathrm{out}}$ is covariant with respect to U. In particular, $\mathcal{A}^{\mathrm{out}}(O_1)$ is invariant under $\mathrm{Ad}\, U(g)$, where $g \in \widetilde{\mathscr{C}}(O_1)$. By definition of $\mathcal{A}^{\mathrm{dir}}$, the two nets $\mathcal{A}^{\mathrm{dir}}$ and $\mathcal{A}^{\mathrm{out}}$ are relatively local.

We saw also in Lemma 4.4 that they generate the same Hilbert subspace $\mathcal{H}^{\mathrm{out}}$. Both nets $\mathcal{A}^{\mathrm{out}}$ and $\mathcal{A}^{\mathrm{dir}}$ are conformal with respect to U, relatively local, and span the same Hilbert subspace. By the standard application of Takesaki's theorem as in Proposition 3.1, these local algebras coincide. □

This concludes our construction. Any conformal net, global or not, contains a free subnet $\mathcal{A}^{\mathrm{out}} = \mathcal{A}^{\mathrm{dir}}$ which generates the massless particle spectrum.

Decoupling of the free field subnet. The next proposition works with Haag dual (for double cones in M) nets with covariance with respect to the Poincaré group. A net has the **split property** if, for each pair $O_1 \subset O_2$ such that $\overline{O_1} \subset O_2$, there is a type I factor $\mathcal{R}$ such that $\mathcal{A}(O_1) \subset \mathcal{R} \subset \mathcal{A}(O_2)$. A **DHR (Doplicher–Haag–Roberts) sector** of the net $\mathcal{A}$ is the equivalence class of a representation π of the global C^*-algebra $\overline{\bigcup_O \mathcal{A}(O)}^{\|\cdot\|}$, where O are double cones under certain conditions [18]. Among others, the most important one is that there is a double cone O such that the restriction of π to $\overline{\bigcup_{O' \perp O} \mathcal{A}(O')}^{\|\cdot\|}$ ($\perp$ denotes the spacelike separation) is unitarily equivalent to the identity representation (the vacuum representation).

PROPOSITION 4.6. *Let $\mathcal{A}$ be a Haag dual subnet of a Haag dual net $\mathcal{F}$ on a separable Hilbert space, and assume that $\mathcal{A}$ has split property and has no nontrivial irreducible DHR sector (if $\mathcal{A} \subset \mathcal{F}$ is an inclusion of conformal nets, we have the Haag duality on $\widetilde{M}$, and we do not need the Haag duality on M). Then $\mathcal{F}$ decouples: namely, $\mathcal{F}(O) = \tilde{\pi}_0(\mathcal{A}(O)) \otimes \mathcal{C}_0(O)$, where $\mathcal{C}(O) = \mathcal{A}(O)' \cap \mathcal{F}(O)$ is the coset net, $\mathcal{C}_0$ is the irreducible vacuum representation of $\mathcal{C}$, and $\tilde{\pi}_0$ is the vacuum representation of $\mathcal{A}$ (the restriction of $\mathcal{A}$ to its cyclic subspace).*

Proof. The argument here is essentially contained in the proof of [14, Theorem 3.4], and has been suggested to apply to globally conformal nets in [2].

The representation of $\mathcal{A}$ on the vacuum Hilbert space of $\mathcal{F}$ is a DHR representation of $\mathcal{A}$ [14, Lemma 3.1] (this can be proved under the split property of $\mathcal{A}$ only, from which it follows that local algebras are properly infinite, and separability of the Hilbert space); hence by the split property it is the direct integral of irreducible representations (see [21, Proposition 56], which is written for nets on S^1 but the arguments apply to nets on M), and by assumption it is the direct sum of copies of the vacuum representation. Hence, on the Hilbert space of $\mathcal{F}$, an element $x \in \mathcal{A}(O)$ is of the form $\tilde{\pi}_0(x) \otimes \mathbb{C}\mathbb{1}$ with an appropriate decomposition $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{K}$. Since $\mathcal{A}$ is Haag dual on its vacuum representation $\tilde{\pi}_0$, we have $\mathcal{A}(O') = \tilde{\pi}_0(\mathcal{A}(O')) \otimes \mathbb{C}\mathbb{1} = \tilde{\pi}_0(\mathcal{A}(O))' \otimes \mathbb{C}\mathbb{1}$. By the relative locality of $\mathcal{F}$ to $\mathcal{A}$, we have $\mathcal{F}(O) \subset \mathcal{A}(O')' = \tilde{\pi}_0(\mathcal{A}(O)) \otimes \mathcal{B}(\mathcal{K})$. Now we have an inclusion

$$\mathcal{A}(O) = \tilde{\pi}_0(\mathcal{A}(O)) \otimes \mathbb{C}\mathbb{1} \subset \mathcal{F}(O) \subset \tilde{\pi}_0(\mathcal{A}(O)) \otimes \mathcal{B}(\mathcal{K}).$$

This relation holds also for a wedge W,

$$\mathcal{A}(W) = \tilde{\pi}_0(\mathcal{A}(W)) \otimes \mathbb{C}\mathbb{1} \subset \mathcal{F}(W) \subset \tilde{\pi}_0(\mathcal{A}(W)) \otimes \mathcal{B}(\mathcal{K}),$$

but the wedge algebra $\tilde{\pi}_0(\mathcal{A}(W))$ in the vacuum representation is a factor [4, 1.10.9 Corollary]. Now, by [17, Theorem A], there is $\mathcal{C}_0(W) \subset \mathcal{B}(\mathcal{K})$ such that

$\mathcal{F}(W) = \tilde{\pi}_0(\mathcal{A}(W)) \otimes \mathcal{C}_0(W)$. It is clear that $\mathcal{F}(W) = \mathcal{A}(W) \vee \mathcal{C}(W)$, where $\mathcal{C}(W) = \mathcal{F}(W) \cap \mathcal{A}(W)'$.

By Haag duality of the both nets $\mathcal{F}$ and $\mathcal{A}$, we have

$$\mathcal{F}(O) = \bigcap_{O\subset W} \mathcal{F}(W) = \bigcap_{O\subset W} \tilde{\pi}_0(\mathcal{A}(W)) \otimes \mathcal{C}_0(W) = \tilde{\pi}_0(\mathcal{A}(O)) \otimes \bigcap_{O\subset W} \mathcal{C}_0(W).$$

By defining $\mathcal{C}(O) := \mathcal{F}(O) \cap \mathcal{A}(O)' = \mathbb{C}\mathbb{1} \otimes \bigcap_{O\subset W} \mathcal{C}_0(W)$ and $\mathcal{C}_0(O) = \bigcap_{O\subset W} \mathcal{C}_0(W)$, we obtain $\mathcal{F}(O) = \tilde{\pi}_0(\mathcal{A}(O)) \otimes \mathcal{C}_0(O) = \mathcal{A}(O) \vee \mathcal{C}(O)$.

If $\mathcal{A} \subset \mathcal{F}$ is an inclusion of conformal nets, we can directly argue with double cones O. Each $\mathcal{A}(O)$ is a factor, the modular group acts geometrically, and Haag duality holds on $\widetilde{M}$ (one should simply transplant the duality argument to $\widetilde{M}$) [7]. □

COROLLARY 4.7. *Let $(\mathcal{A}, U, \Omega)$ be a conformal net, and assume that the massless particle subspace $P_1\mathcal{H}$ of U consists only of the scalar representation with finite multiplicity. Then the free subnet $\mathcal{A}^{\mathrm{out}}$ decouples in $\mathcal{A}$, namely $\mathcal{A}(O) = \mathcal{A}^{\mathrm{out}}(O) \vee \mathcal{C}(O)$, where $\mathcal{C}(O) := \mathcal{A}(O) \cap \mathcal{A}^{\mathrm{out}}(O)'$ is the coset subnet.*

Proof. The scalar free field net has no nontrivial DHR sector [1, 15], and has the split property [8, 13]. These properties are inherited by any finite tensor product. Thus the claim follows from Proposition 4.6. □

5. Open problems

We have shown that massless particles in a conformal net are free. However, massless representations are only one of the families of the irreducible representations of the conformal group. Unfortunately, at the moment the scattering theory, which extracts free fields, is not applicable to the rest of the family. It would be interesting if one could extract other fields by a different device. This would not be very easy, because in general they are expected to be interacting (for example, the super Yang–Mills theory [26]).

As for decoupling, it relies on the split property and the absence of a DHR sector of the scalar free field. As the proofs in the scalar case are based on the arguments in one-particle space and the second quantization, we expect that similar results should hold for each massless finite-helicity representation of the conformal group.

Another interesting question is whether it is possible to prove conformal covariance from scale invariance (under certain additional conditions). Some results have been obtained in this direction [16, 28]. An operator-algebraic proof is unknown (if we do not assume asymptotic completeness, compare with [36]).

By comparing with the result that any massless asymptotically complete model in two dimensions can be obtained by 'twisting' a tensor product net [35, Section 3] [5, Proposition 2.2], one may wonder whether such a structure is available in four dimensions, too. This is not straightforward, because wedges are not suited for the scattering theory in four dimensions. Neither are lightcones, because the intersection of the shifted future and past lightcones does not give back the algebra for a double cone even in the free field net [20]. Related to this issue is whether the S-matrix is a complete invariant of a net under asymptotic completeness. This is open also for massive theories, although partial results are available [6, 27].

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