Scattering in relativistic quantum field theory: basic concepts, tools, and results

Detlev Buchholz[a] and Wojciech Dybalski[b]

[a] Mathematisches Institut, Universität Göttingen
Bunsenstr 3-5, 37073 Göttingen, Germany

[b] Faculty of Mathematics and Computer Science, Adam Mickiewicz University
ul. Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland

Abstract

In this article an overview is provided of basic concepts, tools, and results of quantum field theoretical scattering theory for massive and massless particles. The appearance of infraparticles in theories with long range forces is also discussed as well as the longstanding problem of asymptotic completeness in quantum field theory.

Keywords

Møller operators; scattering matrix; Haag-Ruelle scattering theory; LSZ reduction formulas; particle counters; massless particles; Huygens' principle; infraparticles; asymptotic completeness

Key points

1. Physical ideas and mathematical setting
2. Haag-Ruelle theory
3. LSZ-formalism
4. Asymptotic particle counters
5. Massless particles and Huygens' principle
6. Beyond Wigner's concept of particle
7. Asymptotic completeness

1 Introduction. Physical ideas and mathematical setting

The primary connection of relativistic quantum field theory to experimental physics is through scattering theory, *i.e.* the theory of the collision of elementary (or compound) particles. It is therefore a central topic in quantum field theory and has attracted the attention of leading mathematical physicists. Although a great deal of progress has been made in the mathematically rigorous understanding of the subject, there are important matters which are still unclear, some of which will be outlined below. References to articles in this encyclopedia have an E in front of their number. Other references are given by their numbers only.

In the paradigmatic scattering experiment, several particles, which are initially sufficiently distant from each other, so that the idealization that they do not interact is physically reasonable, approach each other and collide in a region of microscopic extent. The products of this collision then fly apart until they are sufficiently well separated that the approximation of non-interaction is again reasonable. The initial and final states of the objects in the scattering experiment are therefore to be modeled by states of non-interacting, *i.e.* free, fields, which are mathematically represented on Fock space. Typically, what is measured in such experiments is the probability distribution (cross section) for the transitions from a specified state of the incoming particles to a specified state of the outgoing particles.

It should be mentioned that until the late 1950's, the scattering theory of relativistic quantum particles relied upon ideas from non-relativistic quantum mechanical scattering theory (interaction representation, adiabatic limit, *etc*), which were invalid in the relativistic context. Only with the advent of axiomatic quantum field theory did it become possible to properly formulate the concepts and mathematical techniques which will be outlined here.

Scattering theory can be rigorously formulated either in the context of quantum fields satisfying the Wightman axioms [23, E1] or in terms of local algebras satisfying the Haag-Kastler-Araki axioms [15], cf. also [E2]. In brief, the relation between these two settings may be described as follows: In the Wightman setting, the theory is formulated in terms of operator valued distributions ϕ on Minkowski space, the quantum fields, which act on the physical state space. These fields, integrated with test functions f having support in a given region $\mathscr{O}$ of space-time,[1] $\phi(f) = \int d^4x\, f(x)\phi(x)$, form under the operations of addition, multiplication and hermitian conjugation a polynomial *-algebra $\mathscr{P}(\mathscr{O})$ of unbounded operators. In the Haag-Kastler-Araki setting one proceeds from these algebras to algebras $\mathscr{A}(\mathscr{O})$ of bounded operators which, roughly speaking, are formed by the bounded functions A of the operators $\phi(f)$. This step requires some mathematical care, but these subtleties will not be discussed here. As the statements and proofs of the results in these two frameworks differ only in technical details, the theory is presented here in the more convenient setting of algebras of bounded operators (C*-algebras).

Central to the theory is the notion of a particle, which, in fact, is a quite complex concept, the full nature of which is not completely understood, cf. below. In order to maintain the focus on the essential points, we consider in the subsequent sections primarily a single massive particle of integer spin s, *i.e.* a Boson. In standard scattering theory based upon Wigner's characterization, this particle is simply identified with an irreducible unitary representation U_1 of the identity component $\mathscr{P}_+^\uparrow$ of the Poincaré group with spin s and mass $m > 0$. The Hilbert space $\mathscr{H}_1$ upon which $U_1(\mathscr{P}_+^\uparrow)$ acts is called the one-particle space and determines the possible states of a single particle, alone in the universe. Assuming that configurations of several such particles do not interact, one can proceed by a standard construction to a Fock space describing freely propagating multiple particle states,

$$\mathscr{H}_F = \bigoplus_{n \in \mathbb{N}_0} \mathscr{H}_n \,,$$

where $\mathscr{H}_0 = \mathbb{C}$ represents the vacuum and $\mathscr{H}_n$ is the n-fold symmetrized direct product of $\mathscr{H}_1$

[1]We restrict ourselves to four-dimensional Minkowski space. For theories in lower dimensions, cf. [E3].

with itself. This space is spanned by vectors $\Phi_1 \otimes \cdots \otimes \Phi_n$, where $\otimes$ denotes the symmetrized tensor product, representing an n-particle state wherein the k-th particle is in the state $\Phi_k \in \mathscr{H}_1$, $k = 1, \dots n$. The representation $U_1(\mathscr{P}_+^\uparrow)$ induces a unitary representation $U_F(\mathscr{P}_+^\uparrow)$ on $\mathscr{H}_F$ by

$$U_F(\lambda)\,(\Phi_1 \otimes \cdots \otimes \Phi_n) := U_1(\lambda)\Phi_1 \otimes \cdots \otimes U_1(\lambda)\Phi_n\,. \tag{1.1}$$

In interacting theories, the states in the corresponding physical Hilbert space $\mathscr{H}$ do not have such an *a priori* interpretation in physical terms, however. It is the primary goal of scattering theory to identify in $\mathscr{H}$ those vectors which describe, at asymptotic times, incoming, respectively outgoing, configurations of freely moving particles. Mathematically, this amounts to the construction of certain specific isometries (generalized Møller operators) Ω^{in} and Ω^{out} mapping $\mathscr{H}_F$ onto subspaces $\mathscr{H}^{\text{in}} \subset \mathscr{H}$ and $\mathscr{H}^{\text{out}} \subset \mathscr{H}$, respectively, and intertwining the unitary actions of the Poincaré group on $\mathscr{H}_F$ and $\mathscr{H}$. The resulting vectors

$$(\Phi_1 \otimes \cdots \otimes \Phi_n)^{\text{in/out}} := \Omega^{\text{in/out}}\,(\Phi_1 \otimes \cdots \otimes \Phi_n) \in \mathscr{H} \tag{1.2}$$

are interpreted as incoming and outgoing particle configurations in scattering processes.

If, in a theory, the equality $\mathscr{H}^{\text{in}} = \mathscr{H}^{\text{out}}$ holds, then every incoming scattering state evolves, after the collision processes at finite times, into an outgoing scattering state. It is then physically meaningful to define on this space of states the scattering matrix, setting $S = \Omega^{\text{out}\,*}\,\Omega^{\text{in}}$. Physical data such as collision cross sections can be derived from S and the corresponding transition amplitudes $\langle (\Phi_1 \otimes \cdots \otimes \Phi_m)^{\text{in}}, (\Phi'_1 \otimes \cdots \otimes \Phi'_n)^{\text{out}} \rangle$, respectively, by a standard procedure. It should be noted, however, that the physically mandatory equality of the spaces of incoming and outgoing scattering states and the more stringent condition that every state has an interpretation in these terms, *i.e.* $\mathscr{H} = \mathscr{H}^{\text{in}} = \mathscr{H}^{\text{out}}$ (asymptotic completeness), has not yet been established in most models of interest. This problem of asymptotic completeness will be discussed in Sect. 7. It has interesting consequences in the Haag-Kastler-Araki setting, such as the celebrated Bisognano-Wichmann property [E2, 14, 19]. Furthermore, any asymptotically complete quantum field theory can be presented on Fock space such that its underlying unitary representation of the Poincaré group is given by (1.1). In other words, this representation does not contain information about the interaction by itself. It is its action on the local operators which matters.

Before going into details, let us state the few physically motivated postulates entering into the analysis. As discussed, the point of departure is a family of algebras $\mathscr{A}(\mathscr{O})$, more precisely a net, associated with the open subregions $\mathscr{O}$ of Minkowski space and acting on $\mathscr{H}$. Restricting attention to the case of Bosons, we may assume that this net is local in the sense that if $\mathscr{O}_1$ is spacelike separated from $\mathscr{O}_2$, then all elements of $\mathscr{A}(\mathscr{O}_1)$ commute with all elements of $\mathscr{A}(\mathscr{O}_2)$.[2] This is the mathematical expression of the principle of Einstein causality. The unitary representation U of $\mathscr{P}_+^\uparrow$ acting on $\mathscr{H}$ is assumed to satisfy the relativistic spectrum condition (positivity of energy in all Lorentz frames) and, in the sense of equality of sets, $U(\lambda)\mathscr{A}(\mathscr{O})U(\lambda)^{-1} = \mathscr{A}(\lambda\mathscr{O})$ for all $\lambda \in \mathscr{P}_+^\uparrow$ and regions $\mathscr{O}$, where $\lambda\,\mathscr{O}$ denotes the Poincaré transformed region. It is also assumed that the subspace of $U(\mathscr{P}_+^\uparrow)$-invariant vectors is spanned by a single unit vector Ω, representing the vacuum, which has the Reeh-Schlieder property, *i.e.* each set of vectors $\mathscr{A}(\mathscr{O})\Omega$ is dense in $\mathscr{H}$. These standing assumptions will subsequently be amended by further conditions concerning the particle content of the theory.

[2] In the presence of Fermions, these algebras contain also fermionic operators which anti-commute.

2 Haag-Ruelle theory

Haag and Ruelle were the first to establish the existence of scattering states within this general framework, cf. [17]; further substantial improvements are due to Araki and Hepp, cf. [1]. In all of these investigations, the arguments were given for quantum field theories with associated particles (in the Wigner sense) which have strictly positive mass $m > 0$ and for which m is an isolated eigenvalue of the mass operator (upper and lower mass gap). Moreover, it was assumed that states containing the corresponding particle with non-zero probability can be created from the vacuum by local operations. These assumptions allow only for theories with short range interactions and particles carrying localizable charges.

In view of these limitations, Haag-Ruelle theory has been developed in a number of different directions. By now, the scattering theory of massive particles is under complete control, including also particles carrying cone-localizable (gauge or topological) charges in physical spacetime, and the limiting case of particles which are only localizable in wedge-shaped regions, bounded by two characteristic planes. Theories in low dimensions were also discussed, admitting particles with exotic statistics (anyons, plektons). Due to page limitations, these results must go without further mention; we refer the interested reader to the articles [2, 7, 10, E3]. Theories of massless particles and of particles carrying charges of electric or magnetic type (infraparticles) will be discussed in subsequent sections.

We outline here a generalization of Haag-Ruelle scattering theory presented in [12], which covers massive particles with localizable charges without relying on any further constraints on the shape of the mass spectrum. In particular, the scattering of electrically neutral, stable particles fulfilling a sharp dispersion law in the presence of massless particles is included (*e.g.* neutral atoms in their ground states). Mathematically, this assumption can be expressed by the requirement that there exists a subspace $\mathscr{H}_1 \subset \mathscr{H}$ such that the restriction of $U(\mathscr{P}_+^\uparrow)$ to $\mathscr{H}_1$ is a representation of mass $m > 0$. We denote by P_1 the projection in $\mathscr{H}$ onto $\mathscr{H}_1$.

To establish notation, let $\mathscr{O}$ be a bounded spacetime region and let $A \in \mathscr{A}(\mathscr{O})$ be any operator such that $P_1 A\Omega \neq 0$. The existence of such localized (in brief, local) operators amounts to the assumption that the particle carries a localizable charge. That the particle is stable, *i.e.* completely decouples from the underlying continuum states, can be cast into a condition first stated by Herbst: For all sufficiently small $\mu > 0$

$$\|E_\mu (1 - P_1) A\Omega\| \leq c\,\mu^\eta\,, \tag{2.1}$$

for some constants $c, \eta > 0$, where E_μ is the projection onto the spectral subspace of the mass operator corresponding to spectrum in the interval $(m-\mu, m+\mu)$. In the case originally considered by Haag and Ruelle, where m is isolated from the rest of the mass spectrum, this condition is certainly satisfied.

Setting $A(x) := U(x) A U(x)^{-1}$, where $U(x)$ is the unitary implementing the spacetime translation[3] $x = (x_0, \vec{x})$, one puts, for $t \neq 0$,

$$A_t(f) = \int d^4x\, g_t(x_0) f_{x_o}(\vec{x}) A(x)\,. \tag{2.2}$$

Here $x_0 \mapsto g_t(x_0) := g\big((x_0 - t)/|t|^\kappa\big)/|t|^\kappa$ induces a time averaging about t, g being a test function

[3] The velocity of light and Planck's constant are put equal to 1 in what follows

which integrates to 1, whose Fourier transform has compact support, and $1/(1+\eta) < \kappa < 1$ with η as above. The Fourier transform of f_{x_0} is given by $\widetilde{f_{x_0}}(\vec{p}) := \widetilde{f}(\vec{p})e^{-ix_0\omega(\vec{p})}$, where f is some test function on $\mathbb{R}^3$ with $\widetilde{f}(\vec{p})$ having compact support, and $\omega(\vec{p}) = (\vec{p}^2+m^2)^{1/2}$. Note that $(x_0,\vec{x}) \mapsto f_{x_0}(\vec{x})$ is a solution of the Klein-Gordon equation of mass m.

With these assumptions, it follows by a straightforward application of the harmonic analysis of unitary groups that in the sense of strong convergence $A_t(f)\Omega \to P_1 A(f)\Omega$ and $A_t(f)^*\Omega \to 0$ as $t \to \pm\infty$, where $A(f) = \int d^3x\, f(\vec{x})A(0,\vec{x})$. Hence, the limits of the operators $A_t(f)$ may be thought of as creation operators and their adjoints as annihilation operators. These operators are the basic ingredients in the construction of scattering states. Choosing local operators A_k as above and test functions $f^{(k)}$ with disjoint compact supports in momentum space, $k = 1,\dots n$, the scattering states are obtained as limits of the Haag-Ruelle approximants

$$A_{1t}(f^{(1)})\cdots A_{nt}(f^{(n)})\Omega. \tag{2.3}$$

Roughly speaking, the operators $A_{kt}(f^{(k)})$ are localized in spacelike separated regions at asymptotic times t, due to the support properties of the Fourier transforms of the functions $f^{(k)}$. Hence they commute asymptotically because of locality and, by the clustering properties of the vacuum state, the above vector becomes a product of single-particle states. In order to prove convergence one proceeds, in analogy to Cook's method in quantum mechanical scattering theory, to the time derivatives,

$$\begin{aligned} &\partial_t A_{1t}(f^{(1)})\cdots A_{nt}(f^{(n)})\Omega \\ &= \sum_{k\neq l} A_{1t}(f^{(1)})\cdots[\partial_t A_{kt}(f^{(k)}),A_{lt}(f^{(l)})]\cdots A_{nt}(f^{(n)})\Omega \\ &+\sum_k A_{1t}(f^{(1)})\cdots \overset{k}{\vee} \cdots A_{nt}(f^{(n)})\,\partial_t A_{kt}(f^{(k)})\Omega\,, \end{aligned} \tag{2.4}$$

where $\overset{k}{\vee}$ denotes omission of $A_{kt}(f^{(k)})$. Employing techniques of Araki and Hepp, one can prove that the terms in the second line, involving commutators, decay rapidly in norm as t approaches infinity because of locality, as indicated above. By applying condition (2.1) and the fact that the vectors $\partial_t A_{kt}(f^{(k)})\Omega$ do not have a component in the single-particle space $\mathscr{H}_1$, the terms in the third line can be shown to decay in norm like $|t|^{-\kappa(1+\eta)}$. Thus the norm of the vector (2.4) is integrable in t, implying the existence of the strong limits

$$\Big(P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in/out}} := \lim_{t\to\mp\infty} A_{1t}(f^{(1)})\cdots A_{nt}(f^{(n)})\Omega. \tag{2.5}$$

As indicated by the notation, these limits depend only on the single-particle vectors $P_1A_k(f^{(k)})\Omega \in \mathscr{H}_1$, $k = 1,\dots n$, but not on the underlying specific choice of operators and test functions. In order to establish their Fock structure, one employs results on clustering properties of vacuum correlation functions in theories without strictly positive minimal mass. Using this, one can compute inner products of arbitrary asymptotic states and verify that the maps

$$\Big(P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big) \mapsto \Big(P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in/out}} \tag{2.6}$$

extend linearly to isomorphisms $\Omega^{\mathrm{in/out}}$ from the Fock space $\mathscr{H}_F$ onto the subspaces $\mathscr{H}^{\mathrm{in/out}} \subset \mathscr{H}$

generated by the collision states. Moreover, the asymptotic states transform under the Poincaré transformations $U(\mathscr{P}_+^\uparrow)$ as

$$\begin{aligned} &U(\lambda)\,\Big(P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in}/\mathrm{out}} \\ &=\Big(U_1(\lambda)P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes U_1(\lambda)P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in}/\mathrm{out}}. \end{aligned} \tag{2.7}$$

Thus the isomorphisms $\Omega^{\mathrm{in}/\mathrm{out}}$ intertwine the action of the Poincaré group on $\mathscr{H}_F$ and $\mathscr{H}^{\mathrm{in}/\mathrm{out}}$. We summarize these results, which are vital for the physical interpretation of the underlying theory, in the following theorem.

Theorem 1. *Consider a theory of a particle of mass $m > 0$ which satisfies the standing assumptions and the stability condition (2.1). Then there exist canonical isometries $\Omega^{\mathrm{in}/\mathrm{out}}$, mapping the Fock space $\mathscr{H}_F$ based on the single-particle space $\mathscr{H}_1$ onto subspaces $\mathscr{H}^{\mathrm{in}/\mathrm{out}} \subset \mathscr{H}$ of incoming and outgoing scattering states. Moreover, these isometries intertwine the action of the Poincaré transformations on the respective spaces.*

Since the scattering states have been identified with Fock space, asymptotic creation and annihilation operators act on $\mathscr{H}^{\mathrm{in}/\mathrm{out}}$ in a natural manner. This point will be explained in the following section.

3 LSZ formalism

Prior to the results of Haag and Ruelle, an axiomatic approach to scattering theory was developed by Lehmann, Symanzik and Zimmermann (LSZ), based on time-ordered vacuum expectation values of quantum fields [18]. The relative advantage of their approach with respect to Haag-Ruelle theory is that useful reduction formulas for the S-matrix greatly facilitate computations, in particular in perturbation theory. Moreover these formulas are the starting point of general studies of the momentum space analyticity properties of the S-matrix (dispersion relations), cf. [E4]. Within the present general setting, the LSZ method was established by Hepp [16].

For simplicity of discussion, we consider again a single particle type of mass $m > 0$ and integer spin s, subject to condition (2.1). According to the results of the preceding section, one then can consistently define asymptotic creation operators on the scattering states, setting

$$\begin{aligned} &A(f)^{\mathrm{in}/\mathrm{out}}\,\Big(P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in}/\mathrm{out}} \\ &:=\lim_{t\to\mp\infty}A_t(f)\,\Big(P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in}/\mathrm{out}} \\ &=\Big(P_1A(f)\Omega\otimes P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in}/\mathrm{out}}. \end{aligned} \tag{3.1}$$

Similarly, one obtains the corresponding asymptotic annihilation operators,

$$\begin{aligned} &A(f)^{\mathrm{in}/\mathrm{out}\,*}\Big(P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in}/\mathrm{out}} \\ &=\lim_{t\to\mp\infty}A_t(f)^*\,\Big(P_1A_1(f^{(1)})\Omega\otimes\cdots\otimes P_1A_n(f^{(n)})\Omega\Big)^{\mathrm{in}/\mathrm{out}}=0. \end{aligned} \tag{3.2}$$

The preceding two equalities hold if the Fourier transforms of the functions $f, f^{(1)}, \dots f^{(n)}$ have disjoint supports. We mention as an aside that, by replacing the time averaging function g in the definition of $A_t(f)$ by a delta function, the above formulas still hold. But the convergence is then to be understood in the weak Hilbert space topology. In this form the above relations were anticipated by Lehmann, Symanzik and Zimmermann (asymptotic condition).

It is straightforward to proceed from these relations to reduction formulas. Let B be any local operator. Then one has, in the sense of matrix elements between outgoing and incoming scattering states,

$$\begin{aligned} BA(f)^{\text{in}} - A(f)^{\text{out}} B &= \lim_{t\to\infty} \left(BA_{-t}(f) - A_t(f)B\right) \\ &= \lim_{t\to\infty} \left(\int d^4x\, f_{-t}(x) BA(x) - \int d^4x\, f_t(x) A(x) B \right), \end{aligned} \tag{3.3}$$

where $f_t(x) := g_t(x_0) f_{x_0}(\vec{x})$. Due to the (essential) support properties of $f_{\pm t}$, the contributions to the latter two integrals arise, for asymptotic t, from spacetime points x where the localization regions of $A(x)$ and B have a negative timelike (first term), respectively positive timelike (second term) distance. Thus one may proceed from the products of these operators to the time-ordered products $T(BA(x))$, where $T(BA(x)) = A(x)B$ if the localization region of $A(x)$ lies in the future of that of B, and $T(BA(x)) = BA(x)$ if it lies in its past. It is noteworthy that a precise definition of the time ordering for finite x is irrelevant in the present context — any reasonable interpolation between the above relations will do. Similarly, one can define time-ordered products for an arbitrary number of local operators. The preceding limit can then be recast into

$$= \lim_{t\to\infty} \int d^4x\, (f_{-t}(x) - f_t(x)) T(BA(x)). \tag{3.4}$$

The latter expression has a particularly simple form in momentum space. Proceeding to the Fourier transforms of $f_{\pm t}$, one obtains in the limit of large t

$$\left(\widetilde{f_{-t}}(p) - \widetilde{f_t}(p)\right) / \left(p_0 - \omega(\vec{p})\right) \longrightarrow -2\pi i \widetilde{f}(\vec{p})\, \delta(p_0 - \omega(\vec{p})). \tag{3.5}$$

Denoting by $T(B\widetilde{A}(p))$ the Fourier transform of $T(BA(x))$, this implies

$$BA(f)^{\text{in}} - A(f)^{\text{out}} B = -2\pi i \int d^3p\, \widetilde{f}(\vec{p}) \left(p_0 - \omega(\vec{p})\right) T(B\widetilde{A}(-p)) \Big|_{p_0 = \omega(\vec{p})}. \tag{3.6}$$

The restriction of $\left(p_0 - \omega(\vec{p})\right) T(B\widetilde{A}(-p))$ to the manifold $\{p \in \mathbb{R}^4 : p_0 = \omega(\vec{p})\}$ (the "mass shell") is meaningful in the sense of distributions on $\mathbb{R}^3$. By the same token, one obtains

$$A(f)^{\text{out}*} B - BA(f)^{\text{in}*} = -2\pi i \int d^3p\, \overline{\widetilde{f}(\vec{p})} \left(p_0 - \omega(\vec{p})\right) T(\widetilde{A^*}(p) B) \Big|_{p_0 = \omega(\vec{p})}. \tag{3.7}$$

Similar relations, involving an arbitrary number of asymptotic creation and annihilation operators, can be established by analogous considerations. Taking matrix elements of these relations in the vacuum state and recalling the action of the asymptotic creation and annihilation operators on scattering states, one arrives at the following result, which is central in all applications of scattering theory.

Theorem 2. *Consider the theory of a particle of mass $m > 0$ subject to the conditions stated in the preceding sections and let $f^{(1)}, \dots, f^{(n)}$ be any family of test functions whose Fourier transforms have compact and non-overlapping support. Then*

$$
\begin{aligned}
&\big\langle \big(P_1 A_1(f^{(1)})\Omega \otimes \cdots \otimes P_1 A_k(f^{(k)})\Omega\big)^{\mathrm{out}}, \big(P_1 A_{k+1}(f^{(k+1)})\Omega \otimes \cdots \otimes P_1 A_n(f^{(n)})\Omega\big)^{\mathrm{in}}\big\rangle \\
&= (-2\pi i)^n \int \cdots \int d^3p_1 \cdots d^3p_n\, \overline{\widetilde{f^{(1)}}(\vec{p}_1)} \cdots \overline{\widetilde{f^{(k)}}(\vec{p}_k)}\, \widetilde{f^{(k+1)}}(\vec{p}_{k+1}) \cdots \widetilde{f^{(n)}}(\vec{p}_n) \times \\
&\times \prod_{i=1}^{n} \big(p_{i_0} - \omega(\vec{p}_i)\big) \big\langle \Omega, T\big(\widetilde{A_1^*}(p_1) \cdots \widetilde{A_k^*}(p_k) \widetilde{A_{k+1}}(-p_{k+1}) \cdots \widetilde{A_n}(-p_n)\Omega\big)\big\rangle \Big|_{p_{j_0} = \omega(\vec{p}_j)}^{j=1,\dots n},
\end{aligned}
\tag{3.8}
$$

in an obvious notation.

Thus the kernels of the scattering amplitudes in momentum space are obtained by restricting the (by the factor $\prod_{i=1}^{n}(p_{i_0} - \omega(\vec{p}_i))$) amputated Fourier transforms of the vacuum expectation values of the time-ordered products to the positive and negative mass shells, respectively. These are the famous LSZ reduction formulas, which provide a convenient link between the time-ordered (Green's) functions of a theory and its asymptotic particle interpretation.

4 Asymptotic particle counters

The preceding construction of scattering states applies to a significant class of theories; but even if one restricts attention to the case of massive particles, it does not cover all situations of physical interest. For an essential input in the construction is the existence of local operators interpolating between the vacuum and the single-particle states. There may be no such operators at one's disposal, however, either because the particle in question carries a non-localizable charge, or because the given family of operators is too small. The latter case appears, for example, in global and local gauge theories, where in general only the observables are fixed by the principle of gauge invariance, and the physical particle content as well as the corresponding interpolating operators are not known from the outset. As observables create from the vacuum only neutral states, the above construction of scattering states then fails if charged particles are present. Nevertheless, thinking in physical terms, one would expect that the observables contain all relevant information in order to determine the features of scattering states, in particular their collision cross section. That this is indeed the case was first shown by Araki and Haag, cf. [1].

In scattering experiments the measured data are provided by detectors (*e.g.* particle counters) and coincidence arrangements of detectors. Essential features of detectors are their lack of response in the vacuum state and their macroscopic localization. Hence, within the present mathematical setting, a general detector is represented by a positive operator C on the physical Hilbert space $\mathscr{H}$ such that $C\Omega = 0$. Because of the Reeh-Schlieder Theorem, these conditions cannot be satisfied by local operators. However, they can be fulfilled by "almost local" operators. Examples of such operators are easy to produce, putting $C = L^* L$ with

$$L = \int d^4x\, f(x) A(x)\,, \tag{4.1}$$

where A is any local operator and f any test function whose Fourier transform has compact support in the complement of the closed forward lightcone (and hence in the complement of the energy momentum spectrum of the theory). In view of the properties of f and the invariance of Ω

under translations, it follows that $C = L^*L$ annihilates the vacuum and can be approximated with arbitrary precision by local operators. The algebra generated by finite sums and products of these operators C will be denoted by $\mathscr{C}_0$.

When preparing a scattering experiment, the first thing one must do with a detector is to calibrate it, *i.e.* test its response to sources of single-particle states. Within the mathematical setting, this amounts to computing the matrix elements of C in states $\Phi \in \mathscr{H}_1$:

$$\langle \Phi, C\Phi \rangle = \iint d^3p\, d^3q\, \overline{\Phi(\vec{p}\,)}\, \Phi(\vec{q}\,)\, \langle \vec{p}\,|C|\vec{q}\,\rangle\,. \tag{4.2}$$

Note that this single-particle state may be charged. Here $\vec{p} \mapsto \Phi(\vec{p}\,)$ is the momentum space wave function of Φ, $\langle \cdot |C| \cdot \rangle$ is the kernel of C in the single-particle space $\mathscr{H}_1$, and we have omitted (summations over) indices labeling internal degrees of freedom of the particle, if any. The relevant information about C is encoded in its kernel. As a matter of fact, one only needs to know its restriction to the diagonal, $\vec{p} \mapsto \langle \vec{p}\,|C|\vec{p}\,\rangle$. It is called the sensitivity function of C and can be shown to be regular under quite general circumstances, cf. [1, 7].

Given a state $\Psi \in \mathscr{H}$ for which the expectation value $\langle \Psi, C(x)\Psi \rangle$ differs significantly from 0, one concludes that this state deviates from the vacuum in a region about x. For finite x, this does not mean, however, that Ψ has a particle interpretation at x. For that spacetime point may be, for example, the location of a collision center, where two or more particles come close together with non-vanishing probability. Yet if one proceeds to asymptotic times, one expects, in view of the spreading of wave packets, that the probability of finding two or more particles in the same spacetime region is dominated by the single-particle contributions. It is this physical insight which justifies the expectation that the detectors $C(x)$ become particle counters at asymptotic times. Accordingly, one considers for asymptotic t the operators

$$C_t(h) := \int d^3x\, h(\vec{x}/t)\, C(t, \vec{x})\,, \tag{4.3}$$

where h is any test function on $\mathbb{R}^3$. The role of the integral is to sum up all single-particle contributions with velocities in the support of h in order to compensate for the decreasing probability of finding such particles at asymptotic times t about the localization center of the detector. That these ideas are consistent was demonstrated by Araki and Haag, who established the following result [1].

Theorem 3. *Consider, as before, the theory of a massive particle. Let $C^{(1)}, \dots, C^{(n)} \in \mathscr{C}_0$ be any family of detector operators and let $h^{(1)}, \dots, h^{(n)}$ be any family of test functions on $\mathbb{R}^3$. Then, for every state $\Psi^{\text{out}} \in \mathscr{H}^{\text{out}}$ of finite energy,*

$$\begin{aligned} &\lim_{t\to\infty} \left\langle \Psi^{\text{out}}, C_t^{(1)}(h^{(1)}) \cdots C_t^{(n)}(h^{(n)})\, \Psi^{\text{out}} \right\rangle \\ &= \int \cdots \int d^3p_1 \cdots d^3p_n \left\langle \Psi^{\text{out}}, \rho^{\text{out}}(\vec{p}_1) \cdots \rho^{\text{out}}(\vec{p}_n) \Psi^{\text{out}} \right\rangle \prod_{k=1}^{n} h(\vec{p}_k/\omega(\vec{p}_k))\, \langle \vec{p}_k|C^{(k)}|\vec{p}_k\rangle\,. \end{aligned} \tag{4.4}$$

Here $\rho^{\text{out}}(\vec{p}\,)$ is the momentum space density (the product of creation and annihilation operators) of outgoing particles of momentum $\vec{p}$, and (summations over) possible indices labeling internal degrees of freedom of the particle are omitted. An analogous relation holds for incoming scattering states at negative asymptotic times.

This result shows, first of all, that the scattering states have indeed the desired interpretation with regard to the observables, as anticipated in the preceding sections. Since the assertion holds for all scattering states of finite energy, one may replace in the above theorem the outgoing scattering states by any state of finite energy, if the theory is asymptotically complete, *i.e.* $\mathscr{H} = \mathscr{H}^{\mathrm{in}} = \mathscr{H}^{\mathrm{out}}$. Then choosing, in particular, any incoming scattering state and making use of the arbitrariness of the test functions $h^{(k)}$ as well as the knowledge of the sensitivity functions of the detector operators, one can compute the probability distributions of outgoing particle momenta in this state, and thereby the corresponding collision cross sections.

The question of how to construct certain specific incoming scattering states by using only local observables was not settled by Araki and Haag, however. A general method to that effect was outlined in [8]. As a matter of fact, for that method only the knowledge of states in the subspace of neutral states is required. Yet in this approach one would need for the computation of, say, elastic collision cross sections of charged particles the vacuum correlation functions involving at least eight local observables. This practical disadvantage of increased computational complexity of the method is offset by the conceptual advantage of making no appeal to quantities which are *a priori* non-observable.

If one drops the assumption of asymptotic completeness and replaces the vector in equation (4.4) by an arbitrary vector of bounded energy, one may still ask if the Araki-Haag detectors converge. Such convergence results have been obtained by Dybalski and Gérard in [13]. The class of Araki-Haag detectors for which this convergence is under control is quite large: the union of their ranges coincides with the subspace of scattering states (with the vacuum omitted). One may hope that these results are a step towards proving asymptotic completeness, cf. Sect. 7. Let us mention as an aside that the limits of the Araki-Haag detectors correspond to the asymptotic observables in many body quantum mechanics whose existence has been a vital step in proofs of asymptotic completeness [9, Ch. 6.6].

5 Massless particles and Huygens' principle

The preceding general methods of scattering theory apply only to massive particles. Yet taking advantage of the salient fact that massless particles always move with the speed of light, Buchholz succeeded in establishing a scattering theory also for such particles. Moreover, his arguments lead to a quantum version of Huygens' principle [3].

As in the case of massive particles, one assumes that there is a subspace $\mathscr{H}_1 \subset \mathscr{H}$ corresponding to a representation of $U(\mathscr{P}_+^\uparrow)$ of mass $m = 0$ and, for simplicity, integer helicity; moreover, there must exist local operators interpolating between the vacuum and the single-particle states. These assumptions cover, in particular, the important examples of the photon and of Goldstone particles. Picking any suitable local operator A interpolating between Ω and some vector in $\mathscr{H}_1$, one sets, in analogy to (2.2),

$$A_t := \int d^4x\, g_t(x_0)\,(-1/2\pi)\varepsilon(x_0)\,\delta(x_0^2 - \vec{x}^2)\,\partial_0 A(x)\,. \tag{5.1}$$

Here $g_t(x_0) := (1/\ln|t|)\, g\big((x_0 - t)/|\ln t|\big)$ with g as in (2.2), and the solution of the Klein-Gordon equation in (2.2) has been replaced by the fundamental solution of the wave equation (Pauli-Jordan function); furthermore, $\partial_0 A(x)$ denotes the derivative of $A(x)$ with respect to x_0. Then, once again, the strong limit of $A_t\Omega$ as $t \to \pm\infty$ is $P_1 A\Omega$, with P_1 the projection onto $\mathscr{H}_1$.

In order to establish the convergence of A_t as in the LSZ approach, one now uses the fact that these operators are, at asymptotic times t, localized in the complement of some forward, respectively backward, light cone. Because of locality, they therefore commute with all operators which are localized in the interior of the respective cones. More specifically, let $\mathscr{O} \subset \mathbb{R}^4$ be the localization region of A and let $\mathscr{O}_\pm \subset \mathbb{R}^4$ be the two regions having a positive, respectively negative, timelike distance from all points in $\mathscr{O}$. Then, for any operator B which is compactly localized in $\mathscr{O}_\pm$, respectively, one obtains $\lim_{t\to\pm\infty} A_t B\Omega = \lim_{t\to\pm\infty} BA_t\Omega = BP_1 A\Omega$. This relation establishes the existence of the limits

$$A^{\mathrm{in}/\mathrm{out}} = \lim_{t\to\mp\infty} A_t \tag{5.2}$$

on the (by the Reeh-Schlieder property) dense sets of vectors $\{B\Omega : B \in \mathscr{A}(\mathscr{O}_\mp)\} \subset \mathscr{H}$. It requires some more detailed analysis to prove that the limits have all of the properties of a (smeared) free massless field, whose translates $x \mapsto A^{\mathrm{in}/\mathrm{out}}(x)$ satisfy the wave equation and have c-number commutation relations. From these free fields one can then proceed to asymptotic creation and annihilation operators and construct asymptotic Fock spaces $\mathscr{H}^{\mathrm{in}/\mathrm{out}} \subset \mathscr{H}$ of massless particles and a corresponding scattering matrix as in the massive case. The details of this construction can be found in [3], cf. also [15].

It also follows from these arguments that the asymptotic fields $A^{\mathrm{in}/\mathrm{out}}$ of massless particles emanating from a region $\mathscr{O}$, *i.e.* for which the underlying interpolating operators A are localized in $\mathscr{O}$, commute with all operators localized in $\mathscr{O}_\mp$, respectively. This result may be understood as an expression of Huygens' principle. More precisely, denoting by $\mathscr{A}^{\mathrm{in}/\mathrm{out}}(\mathscr{O})$ the algebras of bounded operators generated by those asymptotic fields $A^{\mathrm{in}/\mathrm{out}}$, respectively, one arrives at the following quantum version of Huygens' principle.

Theorem 4. *Consider a theory of massless particles as described above and let $\mathscr{A}^{\mathrm{in}/\mathrm{out}}(\mathscr{O})$ be the algebras generated by massless asymptotic fields $A^{\mathrm{in}/\mathrm{out}}$ with $A \in \mathscr{A}(\mathscr{O})$. Then*

$$\mathscr{A}^{\mathrm{in}}(\mathscr{O}) \subset \mathscr{A}(\mathscr{O}_-)' \quad \textit{and} \quad \mathscr{A}^{\mathrm{out}}(\mathscr{O}) \subset \mathscr{A}(\mathscr{O}_+)' . \tag{5.3}$$

Here the prime denotes the set of bounded operators commuting with all elements of the respective algebras (i.e. their commutants).

6 Beyond Wigner's concept of particle

There is by now ample evidence that Wigner's concept of particle is too narrow in order to cover all particle-like structures appearing in quantum field theory. Examples are the partons which show up in non-abelian gauge theories at very small spacetime scales as constituents of hadrons, but which do not appear at large scales due to the confining forces. Their mathematical description requires a quite different treatment [4], which cannot be discussed here. But even at large scales, Wigner's concept does not cover all stable particle-like systems, the most prominent examples being particles carrying an abelian gauge charge, such as the electron and the proton, which are inevitably accompanied by infinite clouds of ("on shell") massless particles.

The latter problem was discussed first by Schroer, who coined the term *infraparticle* for such systems [21]. Later, Buchholz showed that, as a consequence of Gauss' law, pure states with an abelian gauge charge can neither have a sharp mass nor carry a unitary representation of the

Lorentz group [5], thereby uncovering the simple origin of results found by explicit computations, notably in quantum electrodynamics [22]. Thus one is faced with the question of an appropriate mathematical characterization of infraparticles which generalizes the concept of particle invented by Wigner. Some significant steps in this direction were taken by Fröhlich, Morchio and Strocchi, who based a definition of infraparticles on a detailed spectral analysis of the energy-momentum operators. For an account of these developments and further references, cf. [15].

We outline here an approach, originated by Buchholz, which covers all stable particle-like structures appearing in quantum field theory at asymptotic times. It is based on Dirac's idea of improper particle states with sharp energy and momentum. In the standard (rigged Hilbert space) approach to giving mathematical meaning to these quantities one regards them as vector-valued distributions, whereby one tacitly assumes that the improper states can coherently be superimposed so as to yield normalizable states. This assumption is valid in the case of Wigner particles but fails in the case of infraparticles. A more adequate method of converting the improper states into normalizable ones is based on the idea of acting on them with suitable localizing operators. In the case of quantum mechanics, one could take as a localizing operator any sufficiently rapidly decreasing function of the position operator. It would map the improper "plane wave states" of sharp momentum into finitely localized states which thereby become normalizable. In quantum mechanics, these two approaches can be shown to be mathematically equivalent. The situation is different, however, in quantum field theory.

In quantum field theory, the appropriate localizing operators L are of the form (4.1). They constitute a (non-closed) left ideal $\mathscr{L}$ in the C*-algebra $\mathscr{A}$ generated by all local operators. Improper particle states of sharp energy-momentum p can then be defined as linear maps $|\cdot\rangle_p : \mathscr{L} \to \mathscr{H}$ satisfying[4]

$$U(x)\,|L\rangle_p = e^{ipx}\,|L(x)\rangle_p\,, \quad L \in \mathscr{L}\,. \tag{6.1}$$

In theories of massive particles, one can always find localizing operators $L \in \mathscr{L}$ such that their images $|L\rangle_p \in \mathscr{H}$ are states with a sharp mass. This is the situation covered in Wigner's approach. In theories with long range forces there are, in general, no such operators, however, since the process of localization inevitably leads to the production of low energy massless particles. Yet improper states of sharp momentum still exist in this situation, thereby leading to a meaningful generalization of Wigner's particle concept.

That this characterization of particles covers all situations of physical interest can be justified in the general setting of relativistic quantum field theory as follows. Picking g_t as in (2.2) and any vector $\Psi \in \mathscr{H}$ with finite energy, one can show [6] that the functionals ρ_t, $t \in \mathbb{R}$, given by

$$\rho_t(L^*L) := \int d^4x\, g_t(x_0)\,\langle \Psi, (L^*L)(x)\Psi\rangle\,, \quad L \in \mathscr{L}\,, \tag{6.2}$$

are well defined and form an equicontinuous family with respect to a certain natural locally convex topology on the algebra $\mathscr{C} = \mathscr{L}^*\mathscr{L}$. This family of functionals therefore has, as $t \to \pm\infty$, weak-*-limit points, denoted by σ. The functionals σ are positive on $\mathscr{C}$ but not normalizable. (Technically speaking, they are weights on the underlying algebra $\mathscr{A}$.) Any such σ induces a positive semidefinite scalar product on the left ideal $\mathscr{L}$ given by

$$\langle L_1 \mid L_2\rangle := \sigma(L_1^* L_2)\,, \quad L_1, L_2 \in \mathscr{L}\,. \tag{6.3}$$

[4] It is instructive to (formally) replace here L by the identity operator, making it clear that this relation indeed defines improper states of sharp energy-momentum.

After quotienting out elements of zero norm and taking the completion, one obtains a Hilbert space and a linear map $L \mapsto |L\rangle$ from $\mathscr{L}$ into that space. Moreover, the spacetime translations act on this space by a unitary representation satisfying the relativistic spectrum condition.

It is instructive to compute these functionals and maps in theories of massive particles. Making use of relation (4.4) in Section 4 one obtains, with a slight change of notation,

$$\langle L_1 \mid L_2 \rangle = \int d\mu(p) \, \langle p | L_1^* L_2 | p \rangle \,, \tag{6.4}$$

where μ is a measure giving the probability density of finding at asymptotic times in state Ψ a particle of energy-momentum p. Once again, possible summations over different particle types and internal degrees of freedom have been omitted here. Thus, setting $|L\rangle_p := L\,|p\rangle$, one concludes that the map $L \mapsto |L\rangle$ can be decomposed into a direct integral of improper particle states of sharp energy-momentum, $|\cdot\rangle = \int^{\oplus} d\mu(p)^{1/2}\,|\cdot\rangle_p$. It is crucial that this result can also be established without any *a priori* input about the nature of the particle content of the theory [8, 20], thereby providing evidence of the universal nature of the concept of improper particle states of sharp momentum, as outlined here.

Theorem 5. *Consider a relativistic quantum field theory satisfying the standing assumptions. Then the maps $L \mapsto |L\rangle$ defined above can be decomposed into improper particle states of sharp energy-momentum p,*

$$|\cdot\rangle = \int^{\oplus} d\mu(p)^{1/2}\,|\cdot\rangle_p \,, \tag{6.5}$$

where μ is some measure depending on the underlying state Ψ and the respective time limit taken.

In theories of Wigner particles with equal internal degrees of freedom, the GNS representations induced by pure particle weights $|\cdot\rangle_p$ are mutually unitarily equivalent for different p of the same mass. This property fails for infraparticles, whose “plane wave states" cannot be coherently superimposed, there are no corresponding normalizable states. As was shown by Buchholz, that is the case for electrically charged particle weights, being a consequence of Gauss’s law. The absence of normalizable single electron states was also established in various semi-relativistic models, cf. [E5].

Although a general scattering theory based on improper particle states has not yet been developed, some progress has been made in [8]. There it is outlined how inclusive collision cross sections of scattering states, where an undetermined number of low energy massless particles remains unobserved, can be defined in the presence of long range forces, in spite of the fact that a meaningful scattering matrix may not exist.

7 Asymptotic completeness

Whereas the description of the asymptotic particle features of any relativistic quantum field theory can be based on an arsenal of powerful methods, the question of when such a theory has a complete particle interpretation remains open to date. The only non-trivial (interacting) exceptions are certain two dimensional models with factorizing S-matrices [E3] and theories of wedge-localized particles in higher dimensions [10, 11]; the latter examples are only of moderate interest, however, since they do not exhibit local observables. This situation is in striking contrast to the case of quantum mechanics, where the problem of asymptotic completeness has been completely settled [9].

One may trace the difficulties in quantum field theory back to the possible formation of superselection sectors [15] and the resulting complex particle structures. This feature does not appear in quantum mechanical systems, where the Hilbert space of particle states, including possible bound states, is fixed from the outset. Thus the first step in establishing a complete particle interpretation in a quantum field theory has to be the determination of its full particle content. Here the methods outlined in the preceding section provide a systematic tool since they are based on particle counters, cf. (6.2), by which the charged particle content can be extracted at large times.

From the resulting data, obtained in this manner, one must then reconstruct the full physical Hilbert space of the theory comprising all superselection sectors. For theories in which only massive particles appear, such a construction has been established in [7]. The Hilbert spaces arising from the construction in theories of massive particles contain all scattering states. The question of completeness can then be recast into the familiar problem of the unitarity of the scattering matrix. It is believed that phase space (nuclearity) properties of the theory are of relevance here [15]. This is, in particular, due to the fact that phase space conditions exclude certain models with pathological particle structure, such as generalized free fields with absolutely continuous mass distribution.

However, in theories with long range forces, where a meaningful scattering matrix may not exist, this strategy is bound to fail. Nonetheless, as in most high energy scattering experiments, only some very specific aspects of the particle interpretation are really tested, one may think of other meaningful formulations of completeness. The interpretation of most scattering experiments relies on the existence of conservation laws, such as those for energy and momentum. If a state has a complete particle interpretation, it ought to be possible to fully recover its energy, say, from its asymptotic particle content, *i.e.* there should be no contributions to its total energy which do not manifest themselves asymptotically in the form of particles. Now the mean energy-momentum of a state $\Psi \in \mathscr{H}$ is given by $\langle \Psi, P\Psi \rangle$, P being the energy-momentum operators, and the mean energy-momentum contained in its asymptotic particle content is $\int d\mu(p)\, p$, where μ is the measure appearing in the decomposition (6.5). Hence, in case of a complete particle interpretation the following should hold:

$$\langle \Psi, P\Psi \rangle = \int d\mu(p)\, p\,. \tag{7.1}$$

Similar relations should also hold for other conserved quantities which can be attributed to particles, such as charge, spin *etc*. It seems that such a weak condition of asymptotic completeness suffices for a consistent interpretation of most scattering experiments. One may conjecture that relation (7.1) and its generalizations hold in all theories admitting a local stress energy tensor and local currents corresponding to the charges.

8 Conclusions

In this overview particle aspects of quantum field theory were in focus that arise at asymptotic times. This topic is fundamental for the interpretation of experiments in high energy physics, relying on the concept of collision cross sections. While the construction of asymptotic scattering states of massless and massive particles of Wigner type is under control, there are open questions in the treatment of infraparticles in theories with long rang forces, which are accompanied by infinite clouds of massless particles. Furthermore, the theoretical question is open under which circumstances all states in a theory can be interpreted as asymptotic configurations of particles (problem of asymptotic completeness). The clarification of these questions of physical significance requires further ideas and efforts.

References

[E] The references marked with an E in front of their numbers are articles solicited for this encyclopedia. The titles may change.

[E1] Axiomatic quantum field theory

[E2] Algebraic quantum field theory: objectives, methods, and results

[E3] Construction of two-dimensional models in algebraic quantum field theory

[E4] Dispersion relations

[E5] Infrared problem in quantum field theory

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