The resolvent algebra of non-relativistic Bose fields: observables, dynamics and states

Dedicated to Klaus Fredenhagen on the occasion of his seventieth birthday

Detlev Buchholz

Mathematisches Institut, Universität Göttingen
37073 Göttingen, Germany

Abstract

The gauge-invariant (particle number preserving) observable algebra generated by a non-relativistic Bose field is studied in the C*-algebraic framework of the resolvent algebra. It is shown that this algebra is isomorphic to the inverse limit of a system of approximately finite dimensional C*-algebras. Using this result, it is proven that the algebra is compatible with the Heisenberg picture in the sense that it is stable under the dynamics induced by Hamiltonians involving pair interactions. The argument does not require any approximations, it deals from the outset with the full dynamics. It is outlined how these results shed new light on several topics in many body theory, ranging from causality aspects over the construction of ground states and collision theory up to the determination of thermal equilibrium states. The present approach leads to conceptual simplifications and admits a unified field theoretic treatment of small and large bosonic systems.

Keywords: resolvent algebra, non-relativistic Bose fields, pair potentials, ground states, infra-vacua, collision theory, KMS states

1 Introduction

We continue in this article our study of the resolvent algebras of canonical quantum systems, which provide a natural C*-algebraic framework for the construction of non-trivial dynamics and of the corresponding states [6, 9, 10]. Moreover, they encode characteristic kinematical properties of the underlying finite or infinite quantum systems and have an intriguing algebraic structure [7]. In the present investigation, we supplement these results by a study of the resolvent algebra of a scalar non-relativistic quantum field, obeying canonical commutation relations. This framework provides an efficient alternative to the particle picture, based on position and momentum operators, which is frequently used in the analysis of bosonic many body systems. Let us briefly recall some facts supporting this view.

First, the particle picture generically looses its significance when one is dealing with infinite systems, such as in case of the thermodynamic limit of equilibrium states or in the presence of an infinity of low energy excitations (infrared clouds) in states of finite energy. Strictly speaking, even in systems with a finite particle number the particle interpretation of states acquires significance only at asymptotic times when the interaction between subsystems fades away and particle features emerge. The field theoretic formalism provides a complementary point of view. It is based on the concept of localized operations and observables and thereby tries to model what actually happens in laboratories. Moreover, it simultaneously covers finite and infinite systems and thus provides a uniform basis for their analysis and interpretation.

Second, in order that some algebra generated by field operators may be regarded as a suitable kinematical framework for the formulation of dynamics, it ought to incorporate the solutions of the Heisenberg equations for a large class of Hamiltonians of physical interest. If one considers the polynomial algebra generated by canonical field operators, one finds, however, that it is stable only under a small family of rather trivial dynamics (inducing symplectic transformations). It is less known that this is also the case for the Weyl algebra of exponentials of the fields [9, 15]. The resolvent algebra is much better behaved in this respect. As we shall show, its gauge invariant (particle number preserving) subalgebra is stable under the dynamics induced by a large family of Hamiltonians describing pair interactions. Thus this algebra comprises kinematical observables which can be used at any time to describe the underlying system, independently of the chosen Hamiltonian. In simple words: the algebra does not only contain the initial values but also the solutions of the Heisenberg equations.

Third, in the analysis of infinite systems one frequently relies on finite volume approximations (boxes). This approach requires the consideration of boundary conditions and modifications of kinematical algebras if one wants to proceed to the thermodynamic limit; it is a somewhat cumbersome procedure. In the field theoretic setting of the resolvent algebra one deals from the outset with infinite space. Trapped systems can be described nevertheless by adding to the Hamiltonians external confining potentials. The resulting dynamics still act on the resolvent algebra. Moreover, they converge on the elements of the algebra to the original dynamics if the external potential is turned off. Thus the resolvent algebra provides a convenient framework for the study of infinite systems and their approximations.

In the case considered here, the resolvent algebra is generated by the resolvents of a quantum field which is formed by linear combinations of creation and annihilation operators in s spatial dimensions. These operators satisfy standard canonical commutation relations in position space, informally given by

$$[a(\boldsymbol{x}), a^*(\boldsymbol{y})] = \delta(\boldsymbol{x} - \boldsymbol{y})\, 1\,, \quad [a(\boldsymbol{x}), a(\boldsymbol{y})] = [a^*(\boldsymbol{x}), a^*(\boldsymbol{y})] = 0\,, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^s\,.$$

The advantage gained by using the resolvents of field operators rests upon the fact that, in contrast to the exponential Weyl operators, large values of the fields are suppressed from the outset. Thereby the apparent obstructions to an algebraic treatment of interacting bosonic systems, envisaged for example in [5, Sec. 6.3] and [18], become irrelevant. The simplifications, which arise by using the resolvents, have not yet been exploited in the literature, to the best of our knowledge. Furthermore, this framework can be applied to an arbitrary number and arbitrary types of Bose fields.

As already mentioned, we will consider pair interactions of the field. The generators of the time translations thus have on their standard domains of definition the form

$$H = \int d\boldsymbol{x}\, \boldsymbol{\partial} a^*(\boldsymbol{x})\, \boldsymbol{\partial} a(\boldsymbol{x}) + \int d\boldsymbol{x} \int d\boldsymbol{y}\ a^*(\boldsymbol{x}) a^*(\boldsymbol{y})\, V(\boldsymbol{x} - \boldsymbol{y})\, a(\boldsymbol{x}) a(\boldsymbol{y})\,. \tag{1.1}$$

Here $\boldsymbol{\partial}$ denotes the gradient with regard to $\boldsymbol{x}$ and we assume for simplicty that the potential V is a real, continuous and symmetric function which vanishes at infinity. Singular potentials can be treated by methods outlined in [8, Sec. 6]. We will make substantial use of the fact that H commutes with the particle number operator, given by $N = \int d\boldsymbol{x}\, a^*(\boldsymbol{x}) a(\boldsymbol{x})$.

In the subsequent section we will explicate these structures in more precise mathematical terms. We will make use of the fact that the (abstractly defined) C*-algebra, describing the resolvents of a canonical quantum field, smeared with test functions, is faithfully represented in any regular representation, such as the Fock representation [9, Thm. 4.10]. We will deal with the resolvent algebra in this fixed representation, denoting it by $\mathfrak{R}$, since we can take advantage there of the simple structure of the underlying states.

In the present investigation we focus on the gauge invariant (particle number preserving) subalgebra $\mathfrak{A} \subset \mathfrak{R}$ and determine its structure. This algebra is equipped with a C*-norm and an increasing family of C*-seminorms defining the norm in the limit. It will be convenient in the construction of dynamics to complete the algebra $\mathfrak{A}$ in the locally convex topology induced by the seminorms, leading to a slight extension $\overline{\mathfrak{A}} \supset \mathfrak{A}$. The two algebras coincide (as sets) on all subspaces of Fock space with finite particle number, *i.e.* the extension becomes visible only in states describing an infinity of particles. We will show that the algebra $\overline{\mathfrak{A}}$, called observable algebra in the following, has a comfortable mathematical structure: it is isomorphic to the (bounded) inverse limit of a directed system of approximately finite dimensional C*-algebras. Note that this type of algebras goes by differing names in the literature, cf. [19] and references quoted there.

These observations facilitate the construction of dynamics on the observable algebra $\overline{\mathfrak{A}}$. Relying on arguments given in [7,10], we will show that the unitary time evolution operators e^{itH}, which are fixed by the above Hamiltonians on Fock space, induce by their adjoint action $\boldsymbol{\alpha}(t) \doteq \mathrm{Ad}\, e^{itH}$, $t \in \mathbb{R}$, automorphisms of the faithfully represented algebra $\overline{\mathfrak{A}}$. Moreover, this action is pointwise continuous with regard to time $t \in \mathbb{R}$ and it preserves the quasilocal structure of the observables, thereby complying with a principle of kinematical causality. The proof of this result does not rely on any approximations, it deals from the outset with the full dynamics. The existence of a large family of dynamics is thereby established, showing that the algebra of observables $\overline{\mathfrak{A}}$ is an acceptable kinematical algebra in the sense explained above.

Having settled the framework, we will outline some applications of the formalism to standard problems in many body theory. It is the primary purpose of this part of our article to indicate how the present approach sheds new light on these problems. In particular, we will show how ground states on the algebra $\overline{\mathfrak{A}}$ are constructed, including approximate ground states (infra-vacua) consisting of clouds of low energy Bosons. We will then turn to collision theory and indicate how particle properties are uncoverd at asymptotic times. Our results corroborate the view that the concept of localized field observables is physically meaningful at finite times, whereas particle observables acquire physical significance only at asymptotic times when interactions become negligible. Finally, we indicate how equilibrium states can be constructed in the present setting. Instead of dealing with finite volume approximations, this can be accomplished by considering Hamiltonians with some additional confining potential. They lead to Gibbs states on the algebra $\overline{\mathfrak{A}}$ for the corresponding dynamics, described by density matrices on Fock space. Turning off these external potentials, the automorphic action of the dynamics converges on the observables to its original form and, by compactness arguments, one obtains limit states on $\overline{\mathfrak{A}}$ describing the system in the thermodynamic limit. The mathematical arguments entering in these constructions are based on familiar methods and are largely omitted. They can be elaborated in detail with little additional effort.

Our article is organized as follows. In the subsequent section we recall some basic definitions and facts about the resolvent algebra, which are used in the present invesitigation. Section 3 contains the structural analysis of the algebra of gauge invariant observables generated by the resolvents of a Bose field. The automorphic action of the dynamics on the algebra of observables is established in Sec. 4. In Sec. 5 familiar topics in many body theory are addressed, whereby certain technical points are deferred to an appendix. The article closes with a brief summary and outlook.

2 Framework

We consider representations of the resolvent algebra on Fock space $\mathcal{F} \subset \mathcal{H}$, which is the totally symmetric subspace of the unsymmetrized space $\mathcal{H} = \oplus_{n=0}^{\infty} \mathcal{H}_n$. In more detail, the normalized vacuum vector corresponding to $n = 0$ is denoted by Ω, the single particle space is $\mathcal{H}_1 \simeq L^2(\mathbb{R}^s)$ and the unsymmetrized n-particle space is the n-fold tensor product of the single particle space, $\mathcal{H}_n \simeq L^2(\mathbb{R}^s) \otimes \cdots \otimes L^2(\mathbb{R}^s)$, $n \in \mathbb{N}$. On $\mathcal{H}_n$ acts the unitary representation U_n of the symmetric group $\boldsymbol{\Sigma}_n$, which for $\Psi_1, \dots, \Psi_n \in \mathcal{H}_1$ is given by

$$U_n(\pi)\, \Psi_1 \otimes \cdots \otimes \Psi_n = \Psi_{\pi(1)} \otimes \cdots \otimes \Psi_{\pi(n)} \, , \quad \pi \in \boldsymbol{\Sigma}_n \, .$$

Its mean $\overline{U}_n(\Sigma_n) \doteq (1/n!) \sum_{\pi \in \boldsymbol{\Sigma}_n} U_n(\pi)$ over the group Σ_n is the orthogonal projection in $\mathcal{H}_n$ onto the totally symmetric subspace $\mathcal{F}_n$. We define the n-fold symmetric tensor product of vectors $\Psi_1, \dots, \Psi_n \in \mathcal{H}_1$, putting

$$\Psi_1 \otimes_s \cdots \otimes_s \Psi_n \doteq (1/n!) \sum_{\pi \in \boldsymbol{\Sigma}_n} \Psi_{\pi(1)} \otimes \cdots \otimes \Psi_{\pi(n)} \in \mathcal{F}_n \, .$$

Similarly, we define the symmetric (symmetrized) tensor product of operators $A_1, \ldots, A_n$, which act on $\mathcal{H}_1 = \mathcal{F}_1$, by

$$A_1 \otimes_s \cdots \otimes_s A_n \doteq (1/n!) \sum_{\pi \in \boldsymbol{\Sigma}_n} A_{\pi(1)} \otimes \cdots \otimes A_{\pi(n)} \,.$$

The symmetric subspace $\mathcal{F}_n \subset \mathcal{H}_n$ is stable under the action of these operators, $n \in \mathbb{N}$.

Fields: On Fock space $\mathcal{F}$ there act the creation and annihilation operators a^* and a, which are regularized with complex-valued test functions $f, g \in \mathcal{D}(\mathbb{R}^s)$, having compact support. They satisfy on their standard domain of definition commutation relations given by

$$[a(f), a^*(g)] = \langle f, g \rangle \, \mathbf{1} \,, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0 \,,$$

where $\langle f, g \rangle = \int d\boldsymbol{x} \, \overline{f}(\boldsymbol{x}) g(\boldsymbol{x})$. We recall that $a^*(f)$ is complex linear in f wheras $a(f)$, being the hermitean conjugate of $a^*(f)$, is antilinear in f. This structure can be rephrased in terms of a single real linear, symmetric field operator ϕ given by $\phi(f) \doteq (a^*(f) + a(f))$ with $f \in \mathcal{D}(\mathbb{R}^s)$. It satisfies the commutation relations

$$[\phi(f), \phi(g)] = i\sigma(f, g) \, \mathbf{1} \,, \quad f, g \in \mathcal{D}(\mathbb{R}^s) \,,$$

where $\sigma(f, g) \doteq i(\langle g, f \rangle - \langle f, g \rangle)$ is a non-degenerate real linear symplectic form on $\mathcal{D}(\mathbb{R}^s)$ (being regarded as a real symplectic space). One can recover the creation and annihilation operators from the field operator through the relations

$$2a^*(f) = \phi(f) - i\phi(if) \,, \quad 2a(f) = \phi(f) + i\phi(if) \,, \quad f \in \mathcal{D}(\mathbb{R}^s) \,.$$

The resolvents of the field operators,

$$R(\lambda, f) \doteq (i\lambda + \phi(f))^{-1} \,, \quad \lambda \in \mathbb{R} \backslash \{0\} \,, f \in \mathcal{D}(\mathbb{R}^s) \,,$$

generate, by taking their sums and products and proceeding to the norm closure, the resolvent algebra $\mathfrak{R}$, based on the symplectic space $(\mathcal{D}(\mathbb{R}^s), \sigma)$. As already mentioned, the algebra $\mathfrak{R}$ provides a concrete and faithful representation of the abstractly defined resolvent algebra [9, Thm. 4.10]. We therefore fix it throughout the subsequent analysis.

On Fock space there acts the particle number operator N. It is the generator of the group $\Gamma \simeq U(1)$ of gauge transformations given by

$$e^{isN} \phi(f) e^{-isN} = \phi(e^{is} f) \,, \quad s \in [0, 2\pi] \,, \, f \in \mathcal{D}(\mathbb{R}^s) \,.$$

We denote by $\mathfrak{A} \subset \mathfrak{R}$ the C*-subalgebra of operators, which are invariant under these gauge transformations. It contains for example the resolvents of the operators

$$\big(\phi(f)^2 + \phi(if)^2\big) = 4\, a^*(f)a(f) + 2\langle f, f\rangle\, \mathbf{1}\,, \quad f \in \mathcal{D}(\mathbb{R}^s)\,,$$

which are gauge invariant elements of the resolvent algebra $\mathfrak{R}$. This can be inferred form [10, Prop. 4.1], taking into account that $\sigma(f, if) \neq 0$ if $f \neq 0$, whence $\phi(f)$ and $\phi(if)$ are canonically conjugate operators.

Position and momentum: It will be convenient in our analysis of dynamics to deal also with the quantum mechanical position and momentum operators. These operators are denoted by $\boldsymbol{Q}$, $\boldsymbol{P}$ and satisfy canonical commutation relations which, in an obious notation, are

$$[\boldsymbol{aQ}, \, \boldsymbol{bP}] = i\,(\boldsymbol{ab})\,\mathbf{1}\,, \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^s\,,$$

all other commutators being 0. We make use of the Schrödinger representation of these operators, $\boldsymbol{x} \mapsto (\boldsymbol{aQ}f)(\boldsymbol{x}) = (\boldsymbol{ax})f(\boldsymbol{x})$ and $\boldsymbol{x} \mapsto (\boldsymbol{bP}f)(\boldsymbol{x}) = -i(\boldsymbol{b}\,\boldsymbol{\partial})f(\boldsymbol{x})$ for $f \in L^2(\mathbb{R}^s)$ lying in their respective domains. Given $n \in \mathbb{N}$, we consider pairs of these operators, $\boldsymbol{Q}_1, \boldsymbol{P}_1, \, \dots, \, \boldsymbol{Q}_n, \boldsymbol{P}_n$, which act on the correspondingly numbered components of the unsymmetric n-fold tensor product $\mathcal{H}_n \simeq L^2(\mathbb{R}^s) \otimes \cdots \otimes L^2(\mathbb{R}^s)$ and commute amongst each other. Symmetrized functions of these operators leave the Fock space $\mathcal{F}_n \subset \mathcal{H}_n$ invariant, a prominent example being the restriction of the Hamiltonian in equation (1.1) to $\mathcal{F}_n$,

$$H \upharpoonright \mathcal{F}_n \doteq H_n = \sum_i \boldsymbol{P}_i^2 + \sum_{j \neq k} V(\boldsymbol{Q}_j - \boldsymbol{Q}_k)\,. \tag{2.1}$$

The sums involved here extend over $i, j, k \in \{1, \dots, n\}$. As explained in [9], one can also define resolvent algebras of position and momentum operators, but we make no use of this formalism here.

3 Structure of observables

In this section we will clarify the structure of the gauge-invariant subalgebra $\mathfrak{A}$ of the resolvent algebra of Bose fields. In a first step we determine the properties of special elements of this algebra.

Lemma 3.1. *Let $M = \prod_{k=1}^m R(\lambda_k, f_k) \in \mathfrak{R}$ be any ordered product (monomial) of resolvents of the field, $\lambda_k \in \mathbb{R}\backslash\{0\}$, $f_k \in \mathcal{D}(\mathbb{R}^s)\backslash\{0\}$, $k = 1, \dots, m$. Its mean over the gauge* *group* $\overline{M}^{\Gamma} \doteq (2\pi)^{-1} \int_0^{2\pi} dt\, e^{itN} M e^{-itN}$*, defined in the strong operator topology, is an element of* $\mathfrak{A}$*. Moreover, denoting by* L *the complex linear span generated by* $(f_1, \dots, f_m)$ *and by* $\mathcal{F}(L) \subset \mathcal{F}$ *the Fock space based on the subspace* $L \subset L^2(\mathbb{R}^s)$*, the restriction* $\overline{M}^{\Gamma} \upharpoonright \mathcal{F}(L)$ *is a compact operator.*

Proof. Noticing that the space (L, σ) is a finite dimensional non-degenerate symplectic subspace of $(\mathcal{D}(\mathbb{R}^s), \sigma)$, let $\mathfrak{R}(L) \subset \mathfrak{R}$ be the resolvent algebra generated by the resolvents $R(\lambda, h)$, where $\lambda \in \mathbb{R}\backslash\{0\}$, $h \in L$. This algebra acts irreducibly on $\mathcal{F}(L)$ and one has $e^{isN} M e^{-isN} \in \mathfrak{R}(L)$, $s \in [0, 2\pi]$. Consider now the function

$$s, t \mapsto e^{isN} M^* e^{-isN} e^{itN} M e^{-itN}, \quad s, t \in [0, 2\pi].$$

Since $e^{itN} R(\lambda, h) e^{-itN} = R(\lambda, e^{it} h)$, and similarly for the adjoints, the values of this function lie in the intersection of the principal ideals in $\mathfrak{R}(L)$, which are generated by the individual gauge-transformed resolvents in the above product. According to [7, Prop. 4.4], this intersection coincides with the principal ideal generated by the reordered product

$$R(\lambda_1, e^{is} f_1)^* R(\lambda_1, e^{it} f_1) \cdots R(\lambda_m, e^{is} f_m)^* R(\lambda_m, e^{it} f_m).$$

But the latter operator acts as a compact operator on $\mathcal{F}(L)$ if all adjacent pairs of resolvents are generated by canonically conjugate operators, *i.e.* if $\sigma(e^{is} f_k, e^{it} f_k) \neq 0$ for $k = 1, \dots, m$, cf. [9, Thm. 5.4]. So the above function has, for almost all $(s, t) \in [0, 2\pi] \times [0, 2\pi]$, values in the compact operators on $\mathcal{F}_L$ and is bounded. Hence the double integral (defined in the strong operator topology on $\mathcal{F}_L$)

$$\overline{M}^{\Gamma *}\, \overline{M}^{\Gamma} = \int_0^{2\pi} ds \int_0^{2\pi} dt\, e^{isN} M^* e^{-isN} e^{itN} M e^{-itN}$$

is a compact operator as well. Taking its square root and performing a polar decomposition, we find that $\overline{M}^{\Gamma} \upharpoonright \mathcal{F}(L)$ is compact and consequently an element of the compact ideal of $\mathfrak{R}(L) \upharpoonright \mathcal{F}(L)$, cf. [9, Thm. 5.4]. Since $\mathfrak{R}(L)$ is faithfully represented on $\mathcal{F}(L)$, we conclude that $\overline{M}^{\Gamma} \in \mathfrak{R}(L) \subset \mathfrak{R}$. Moreover, $\overline{M}^{\Gamma}$ commutes by construction with the gauge transformations which shows that $\overline{M}^{\Gamma} \in \mathfrak{A}$. $\square$

Every element of $\mathfrak{R}$ and hence *a fortiori* of $\mathfrak{A}$ can be approximated in norm by sums of monomials of resolvents and the unit operator. It therefore follows from Lemma 3.1 that the finite sums of operators of the form $\sum_i c_i \overline{M_i}^{\Gamma}$ with $\overline{M_i}^{\Gamma}$ as in the lemma, including the unit operator $\mathbf{1}$, are norm dense in $\mathfrak{A}$. In fact, given any $A \in \mathfrak{A}$ and $\varepsilon > 0$ there exists a sum of monomials $\sum_i c_i M_i$ such that $\| \sum_i c_i M_i - A\| < \varepsilon$. Since A is gauge invariant, we obtain, taking a mean over the gauge group, $\| \sum_i c_i \overline{M_i}^{\Gamma} - A\| < \varepsilon$, as claimed. Thus we conclude that the algebra $\mathfrak{A}$ is generated by the unit operator and the gauge invariant operators in the minimal compact ideals of all subalgebras of $\mathfrak{R}(L) \subset \mathfrak{R}$ corresponding to the finite dimensional non-degenerate symplectic subspaces $(L, \sigma) \subset (\mathcal{D}(\mathbb{R}^s), \sigma)$.

In the next step we analyze in detail the structure of the restrictions of the algebra $\mathfrak{A}$ to the subspaces $\mathcal{F}_n \subset \mathcal{F}$ of particle number $n \in \mathbb{N}$. To this end we introduce the following quantities: given $1 \leq m \leq n$, let $\mathfrak{C}_m$ be the C*-algebra of compact operators on $\mathcal{F}_m$. This algebra coincides with the unique m-fold symmetric (symmetrized) tensor product of the algebra of compact operators on $\mathcal{F}_1$,

$$\mathfrak{C}_m = \underbrace{\mathfrak{C}_1 \otimes_s \cdots \otimes_s \mathfrak{C}_1}_{m} .$$

For $m = 0$ we put $\mathfrak{C}_0 = \mathbb{C}\,1$. We embed these algebras into the algebra of bounded operators on $\mathcal{F}_n$, putting

$$\mathfrak{C}_{m,n} \doteq \mathfrak{C}_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m}, \quad m \leq n \,. \tag{3.1}$$

In particular, $\mathfrak{C}_{0,n} = \mathbb{C}\,\mathbf{1}$ and $\mathfrak{C}_{n,n} = \mathfrak{C}_n$. These embeddings are isometric, *i.e.*

$$\|C \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m} \|_n = \|C\|_m \,, \quad C \in \mathfrak{C}_m \,,\ 0 \leq m \leq n \,,$$

where $\| \cdot \|_n$ denotes the operator norm on $\mathcal{F}_n$, $n \in \mathbb{N}_0$. Conversely, the algebra $\mathfrak{C}_{m,n-1}$ can be recovered from $\mathfrak{C}_{m,n}$, $m < n$, by omitting from its respective elements a tensor factor 1. With these ingredients we can state the following definition.

Definition: Let $n \in \mathbb{N}_0$. The algebra $\mathfrak{K}_n$ is the unital C*-algebra on $\mathcal{F}_n$ that is generated by $\mathfrak{C}_{m,n}$, $0 \leq m \leq n$. It coincides with the linear span of $\mathfrak{C}_{m,n}$, $0 \leq m \leq n$. Note that the algebras $\mathfrak{K}_n$ are AF-algebras, *i.e.* approximately finite dimensional, since the algebra of compact operators is of this type. The canonical embeddings (positive maps) of these algebras into each other are denoted by $\epsilon_n : \mathfrak{K}_n \to \mathfrak{K}_{n+1}$. They are defined by

$$\epsilon_n(K_n) = K_n \otimes_s 1 \,, \quad K_n \in \mathfrak{K}_n \,.$$

Thus the family $(\mathfrak{K}_n, \epsilon_n)_{n \in \mathbb{N}_0}$ constitutes a directed system of C*-algebras. The left inverses $\kappa_n : \mathfrak{K}_n \to \mathfrak{K}_{n-1}$ of these embeddings are contracting maps which act on the subspaces

$\mathfrak{C}_{m,n} \subset \mathfrak{K}_n$ according to

$$\kappa_n(C_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes 1}_{n-m}) = (1 - \delta_{m\,n})\ (C_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m-1})\,, \quad 0 \leq m < n\,.$$

For $n = 0$ we put $\kappa_0 = 0$. The family $(\mathfrak{K}_n, \kappa_n)_{n \in \mathbb{N}_0}$ constitutes an inverse (downwards directed) system. We also consider the maps $\kappa_{m,n} : \mathfrak{K}_n \to \mathfrak{K}_m$ given by $\kappa_{m,n} \doteq \kappa_{m+1} \circ \cdots \circ \kappa_n$, $0 \leq m \leq n-1$, where $\kappa_{n-1,n} \doteq \kappa_n$.

After these preparations we can establish the following fact, making use of the preceding lemma.

Lemma 3.2. *Let $n \in \mathbb{N}_0$. Then $\mathfrak{A} \upharpoonright \mathcal{F}_n \subseteq \mathfrak{K}_n$.*

Proof. As already explained, the elements of $\mathfrak{A}$ can be approximated in the norm topology by sums of gauge-averaged monomials $\overline{M}^\Gamma$. So it suffices to show that $\overline{M}^\Gamma \upharpoonright \mathcal{F}_n \in \mathfrak{K}_n$ for any such operator.

Let $L \subset L^2(\mathbb{R}^s)$ be the complex subspace generated by the test functions appearing in the resolvents, which are factors of the given monomial M, cf. Lemma 3.1. This space determines the resolvent algebra $\mathfrak{R}(L)$, the Fock space $\mathcal{F}(L)$ and the particle number operator $N(L)$ acting on it. Since L is finite dimensional, the resolvent of $N(L)$ is a compact operator on $\mathcal{F}(L)$ and hence belongs to the compact ideal of $\mathfrak{R}(L)$, cf. [9, Thm. 5.4]. As a matter of fact, since this resolvent is also gauge invariant, it belongs to $\mathfrak{A}(L)$ and the same is true for its finite dimensional spectral projections $E_m(L)$ corresponding to the spectral values $m \in \mathbb{N}_0$. We decompose $\overline{M}^\Gamma$ into operators of finite rank, $\overline{M}^\Gamma = \sum_{m=0}^\infty \overline{M}^\Gamma E_m(L)$. The sum converges in norm in view the compactness properties of $\overline{M}^\Gamma$ on $\mathcal{F}(L)$, proven in the preceding lemma.

We can determine now the action of the operators $\overline{M}^\Gamma E_m(L)$, $m \in \mathbb{N}_0$, on $\mathcal{F}_n$. To this end we decompose $\mathcal{F}_n$ into the orthogonal sum of symmetric tensor products

$$\mathcal{F}_n = \sum_{k=0}^n \underbrace{L \otimes_s \cdots \otimes_s L}_{k} \otimes_s \underbrace{L^\perp \otimes_s \cdots \otimes_s L^\perp}_{n-k}\,,$$

where $L^\perp$ denotes the orthogonal complement of L in $L^2(\mathbb{R}^s)$. Since the elements of $\mathfrak{A}(L)$ commute with the creation operators $a^*(h^\perp)$, $h^\perp \in L^\perp$, the operators in $\mathfrak{A}(L)$ act non-trivially (*i.e.* differ from a multiple of the identity) only on factors in the tensor products which are contained in L. Thus, taking also into account the action of the projections $E_m(L) \in \mathfrak{A}(L)$, $m \in \mathbb{N}_0$, on the subspaces $\mathcal{F}_k(L) \subset \mathcal{F}(L)$, $k \in \mathbb{N}_0$, we obtain for $m \leq n$

$$\overline{M}^\Gamma E_m(L) \upharpoonright \mathcal{F}_n = \big(\overline{M}^\Gamma E_m(L) \upharpoonright \underbrace{L \otimes_s \cdots \otimes_s L}_{m}\big) \otimes_s \underbrace{L^\perp \otimes_s \cdots \otimes_s L^\perp}_{n-m}\,,$$

where $\overline{M}^{\Gamma} E_0(L) \in \mathbb{C}\,\mathbf{1}$. If $m > n$ we have $\overline{M}^{\Gamma} E_m(L) \upharpoonright \mathcal{F}_n = 0$.

Now, given $m \in \mathbb{N}_0$, one clearly has $E_m(L)\mathcal{F}_m \subset \mathcal{F}_m(L)$. Hence $\overline{M}^{\Gamma} E_m(L)$ acts as a compact (even finite rank) operator C_m on $\mathcal{F}_m$ according to Lemma 3.1. Denoting by F_L the finite dimensional projection onto $L \subset L^2(\mathbb{R}^s)$, the preceding step therefore implies that for any $m \leq n$

$$\overline{M}^{\Gamma} E_m(L) \upharpoonright \mathcal{F}_n = (C_m \underbrace{F_L \otimes_s \cdots \otimes_s F_L}_{m}) \otimes_s \underbrace{(1 - F_L) \otimes_s \cdots \otimes_s (1 - F_L)}_{n-m} \in \mathfrak{K}_n \, .$$

Since $\overline{M}^{\Gamma} \upharpoonright \mathcal{F}_n = \big(\sum_{m=0}^{n} \overline{M}^{\Gamma} E_m(L) \big) \upharpoonright \mathcal{F}_n$, this completes the proof of the statement. $\square$

We show next that $\mathfrak{A} \upharpoonright \mathcal{F}_n$ actually coincides with $\mathfrak{K}_n$. Moreover, we identify operators which conform to a certain degree with the embeddings of these spaces into each other, $n \in \mathbb{N}_0$.

Lemma 3.3. *Let $n \in \mathbb{N}_0$. Then $\mathfrak{A} \upharpoonright \mathcal{F}_n = \mathfrak{K}_n$. Moreover, given $m \in \mathbb{N}_0$ and $A \in \mathfrak{A}$ such that $A \upharpoonright \mathcal{F}_m = K_m \in \mathfrak{K}_m$, there exist for any $M \geq m$ operators $A_M \in \mathfrak{A}$ such that*

$$A_M \upharpoonright \mathcal{F}_n = K_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m}, \quad m \leq n \leq M \, .$$

Remark: Since $\mathfrak{K}_n$ contains the compact operators on $\mathcal{F}_n$, the algebra $\mathfrak{A}$ acts irreducibly on this space as a consequence of this lemma. Yet, whereas $\mathfrak{A}$ acts faithfully on $\mathcal{F}$ (since $\mathfrak{R}$ does), its action on the subspaces $\mathcal{F}_n \subset \mathcal{F}$ is not faithful. We denote by $\mathfrak{J}_n \subset \mathfrak{A}$ the closed two-sided ideal of operators that are annihilated on $\mathcal{F}_n$ and show below that these ideals are nested, *i.e.* $\mathfrak{J}_{n+1} \subset \mathfrak{J}_n$. Hence the seminorms $\| \cdot \|_n$ on $\mathfrak{A}$ are increasing, $n \in \mathbb{N}_0$, and $\| \cdot \| \doteq \lim_{n \to \infty} \| \cdot \|_n$ agrees with the C*-norm on $\mathfrak{A}$.

Proof. Since the quotient C*-algebra $\mathfrak{A}/\mathfrak{J}_n$ is faithfully represented on $\mathcal{F}_n$, it suffices for the proof of the statement to show that $\mathfrak{A} \upharpoonright \mathcal{F}_n$ is dense in $\mathfrak{K}_n$ with regard to the operator norm on $\mathcal{F}_n$. We will accomplish this by showing that for each $0 \leq m \leq n$ the algebra $\mathfrak{A} \upharpoonright \mathcal{F}_n$ includes all operators in $\mathfrak{C}_{m,n} \subset \mathfrak{K}_n$ that arise from operators in $\mathfrak{C}_m$ having finite rank.

For $m = 0$ we have $\mathfrak{C}_{0,n} = \mathbb{C}\,\mathbf{1}$, so there is nothing to prove. Next, let $m = 1$ and let $\{e_j \in \mathcal{D}(\mathbb{R}^s)\}_{j \in \mathbb{N}}$ be any orthonormal basis in $L^2(\mathbb{R}^s) \simeq \mathcal{F}_1$. The algebra $\mathfrak{C}_1$ contains in particular the matrix units corresponding to this basis and their linear span is norm dense in $\mathfrak{C}_1$. The action of these matrix units on the vectors $\Phi_1 \in \mathcal{F}_1$ is given by

$$M_{ik} \Phi_1 = \langle e_k, \Phi_1 \rangle \, e_i \, , \quad i, k \in \mathbb{N} \, .$$

We recall that their embeddings into $\mathfrak{C}_{1,n}$ are defined by the symmetric (symmetrized) tensor product $M_{ik} \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-1}$, $i,k \in \mathbb{N}$.

In order to establish the existence of operators in $\mathfrak{A}$ that induce the same action on $\mathcal{F}_n$, we consider the complex rays $L_j \doteq \mathbb{C}\, e_j$ and the corresponding particle number operators $N(L_j) = a^*(e_j)a(e_j)$ on the Fock spaces $\mathcal{F}(L_j)$, $j \in \mathbb{N}$. As already explained, all spectral projections of $N(L_j)$ corresponding to spectral values in the finite interval $[0, M]$ are finite dimensional for any $M \geq n$ and thus are elements of the compact ideal of the resolvent algebra $\mathfrak{R}(L_j)$. Denoting the sum of these projections by $E_{[0,M]}(L_j)$, we therefore have

$$X_M(L_j) \doteq E_{[0,M]}(L_j)\,(1 + N(L_j))^{-1/2}\, a(e_j) \;\in \mathfrak{R}(L_j) \subset \mathfrak{R}\,, \quad j \in \mathbb{N}_0\,.$$

The resulting operators $W_M(i,k) \doteq X_M(L_i)^* X_M(L_k)$ are gauge invariant, hence elements of $\mathfrak{A}$, and a routine computation shows that for any $M \geq n$

$$W_M(i,k) \upharpoonright \mathcal{F}_n = M_{ik} \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-1}\,, \quad i,k \in \mathbb{N}\,.$$

Since all operators in $\mathfrak{C}_1$ of finite rank can be decomposed into finite sums of matrix units, this proves the statement for $m = 1$.

Given $n \in \mathbb{N}$, the operators $W_M(i,k)$, $M \geq n$, can now be used in order to obtain operators of finite rank in the spaces $\mathfrak{C}_{m,n}$ for any $m \leq n$. One proceeds from the equality

$$\begin{aligned} &W_M(i_1,k_1)\cdots W_M(i_m,k_m) \upharpoonright \mathcal{F}_n \\ &\qquad = (M_{i_1k_1} \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-1}) \cdots (M_{i_mk_m} \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-1})\,. \end{aligned}$$

Because of the symmetrisation, the right hand side of this equality is a linear combination of operators in $\mathfrak{C}_{m',n}$ that arise from matrix units in $\mathfrak{C}_{m'}$, $0 \leq m' \leq m$. By an obvious recursive procedure (based on normal ordering) one can determine certain specific linear combinations of these operators of the form

$$W_M(i_1,\dots i_m,k_1,\dots k_m) \doteq \sum c_{j_1,\dots j_{m'},k_1,\dots k_{m'}} W_M(j_1,l_1)\cdots W_M(j_{m'},k_{m'}) \in \mathfrak{A},$$

where the sum extends over $j_1,\dots,j_{m'} \in \{i_1,\dots i_m\}$, $l_1,\dots,l_{m'} \in \{k_1,\dots k_m\}$ and $m' \leq m$. With properly determined coefficents, these operators satisfy

$$W_M(i_1,\dots i_m,k_1,\dots k_m) \upharpoonright \mathcal{F}_n = M_{i_1k_1} \otimes_s \cdots \otimes_s M_{i_m,k_m} \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m}\,.$$

Hence $\mathfrak{A} \restriction \mathcal{F}_n$ contains for any given $0 \leq m \leq n$ all operators in $\mathfrak{C}_{m,n}$ that arise from operators in $\mathfrak{C}_m$ of finite rank. Recalling that these operators are norm dense in the compact operators, we conclude that $\mathfrak{A} \restriction \mathcal{F}_n$ is norm dense in $\mathfrak{K}_n$ with regard to the operator norm on $\mathcal{F}_n$. Bearing in mind the initial remarks, this completes the proof of the first part of the statement.

As to the second part of the statement, the operators $W_M(i_1, \dots i_m, k_1, \dots k_m) \in \mathfrak{A}$, constructed in the preceding step, give rise by their action on $\mathcal{F}_n$ to embeddings of the matrix units $M_{i_1 k_1} \otimes_s \cdots \otimes_s M_{i_m, k_m} \in \mathfrak{C}_m$ into the spaces $\mathfrak{C}_{m,n}$, $m \leq n \leq M$. Bearing in mind the monotonicity of the seminorms on $\mathfrak{A}$, the proof of the statement is completed by applying the above density argument to finite sums of the operators $W_M(i_1, \dots i_m, k_1, \dots k_m) \restriction \mathcal{F}_M$. □

The preceding result shows that the elements $A \in \mathfrak{A}$ determine special elements of the directed system $(\mathfrak{K}_n, \epsilon_n)_{n \in \mathbb{N}_0}$. In order to better understand their global structure, we make use of locality properties of $\mathfrak{A}$ and clustering properties of the states in $\mathcal{F}$. To this end we consider vectors in $\mathcal{F}$, where (some of the) single particle components undergo large spatial translations. We adopt the following notation: given $k \in \mathbb{N}$ and $\Phi_1, \dots, \Phi_k \in \mathcal{F}_1$, we put

$$\boldsymbol{\Phi}^k(\boldsymbol{x}) \doteq e^{i\boldsymbol{x}\boldsymbol{P}} \Phi_1 \otimes_s \cdots \otimes_s e^{i\boldsymbol{x}\boldsymbol{P}} \Phi_k \in \mathcal{F}_k \,, \quad \boldsymbol{\Phi}^k(0) \doteq \boldsymbol{\Phi}^k \,.$$

Considering the tensor product of translated and untranslated vectors

$$\boldsymbol{\Phi}^m \otimes_s \boldsymbol{\Phi}^{n-m}(\boldsymbol{x}) \in \mathcal{F}_n \quad 1 < m < n, \, n \in \mathbb{N} \,,$$

the following result obtains.

Lemma 3.4. *Let $A \in \mathfrak{A}$ and $n \in \mathbb{N}$ be given.*
(i) If $0 < m < n$, one has

$$\lim_{\boldsymbol{x} \to \infty} \langle \boldsymbol{\Psi}^m \otimes_s \boldsymbol{\Psi}^{n-m}(\boldsymbol{x}), \, A \; \boldsymbol{\Phi}^m \otimes_s \boldsymbol{\Phi}^{n-m}(\boldsymbol{x}) \rangle = \langle \boldsymbol{\Psi}^m, \, A \, \boldsymbol{\Phi}^m \rangle \; \langle \boldsymbol{\Psi}^{n-m}, \boldsymbol{\Phi}^{n-m} \rangle \,.$$

If $m = 0$, one has

$$\lim_{\boldsymbol{x} \to \infty} \langle \boldsymbol{\Psi}^n(\boldsymbol{x}), \, A \; \boldsymbol{\Phi}^n(\boldsymbol{x}) \rangle = \langle \Omega, \, A \, \Omega \rangle \; \langle \boldsymbol{\Psi}^n, \boldsymbol{\Phi}^n \rangle \,,$$

where $\Omega \in \mathcal{F}_0$ is the Fock vacuum.
(ii) Let $A \restriction \mathcal{F}_n = K_n \in \mathfrak{K}_n$, then $A \restriction \mathcal{F}_m = \kappa_{m,n}(K_n)$, $0 \leq m \leq n-1$. Here $\kappa_{m,n}$ are the contracting (inverse) maps on the directed system $\{\mathfrak{K}_n, \epsilon_n\}_{n \in \mathbb{N}}$, defined above.

Remark: It follows from this lemma that any element $A \in \mathfrak{A}$ which annihilates all vectors in $\mathcal{F}_n$ also annihilates all vectors in $\mathcal{F}_m$, $0 \leq m \leq n$. This entails the inclusion of ideals $\mathfrak{J}_{n+1} \subset \mathfrak{J}_n$, $n \in \mathbb{N}_0$, which was mentioned above.

Proof. (i) The first statement is a well known consequence of the canonical commutation relations. In the case at hand, these relations imply

$$[R(\lambda, f), a^*(e^{i\boldsymbol{x}\boldsymbol{P}} g)] = \langle f, e^{i\boldsymbol{x}\boldsymbol{P}} g\rangle \, R(\lambda, f)^2 \,, \quad f, g \in \mathcal{D}(\mathbb{R}^s)\,.$$

Since $e^{i\boldsymbol{x}\boldsymbol{P}} \to 0$ weakly on $L^2(\mathbb{R}^s)$ for $\boldsymbol{x} \to \infty$, one can commute in this limit all creation operators, used in creating the vectors $\boldsymbol{\Phi}^{n-m}(\boldsymbol{x})$ from the vacuum, to the left of any given monomial M in the resolvents. Applying their adjoints to the vectors $\boldsymbol{\Psi}^m \otimes_s \boldsymbol{\Psi}^{n-m}(\boldsymbol{x})$, the statement then follows in this special case. Since the finite sums of monomials are norm-dense in $\mathfrak{A}$, this result extends to all $A \in \mathfrak{A}$.

(ii) Let $A \upharpoonright \mathcal{F}_n = K_n = C_{0,n} + C_{1,n} + \cdots + C_{n,n}$, where $C_{j,n} \in \mathfrak{C}_{j,n}$, $0 \leq j \leq n$. Replacing A in the matrix elements in (i) by this sum of operators, we obtain terms of the form

$$\langle \boldsymbol{\Psi}^m \otimes_s \boldsymbol{\Psi}^{n-m}(\boldsymbol{x}),\, C_{j,n}\; \boldsymbol{\Phi}^m \otimes_s \boldsymbol{\Phi}^{n-m}(\boldsymbol{x})\rangle\,, \quad 0 \leq j \leq n\,.$$

Bearing in mind that the algebra $\mathfrak{C}_j$ coincides with the j-fold symmetric C*-tensor product of the algebra of compact operators $\mathfrak{C}_1$ on $\mathcal{F}_1$, it suffices to consider the case where the operator $C_{j,n}$ is the n-fold symmetric tensor product of j compact operators and $n - j$ unit operators. Now if $j > m$, then at least one of the compact operators in $C_{j,n}$ acts on a single particle component of $\boldsymbol{\Phi}^{n-m}(\boldsymbol{x})$. This component converges weakly to 0 in the limit of large $\boldsymbol{x}$. But compact operators map weakly convergent sequences of vectors into strongly convergent sequences, so it follows that the matrix elements of $C_{j,n}$ vanish in this limit if $j > m$. If $j \leq m$, all contributions arising from the action of compact operators on $\boldsymbol{\Phi}^{n-m}(\boldsymbol{x})$ vanish in this limit as well. In other words, only those contributions survive where all j compact components in $C_{j,n}$ act on $\boldsymbol{\Phi}^m$. The remaining unit operators do not affect the vector $\boldsymbol{\Phi}^{n-m}(\boldsymbol{x})$. Thus we obtain

$$\begin{aligned}&\lim_{\boldsymbol{x}\to\infty} \langle \boldsymbol{\Psi}^m \otimes_s \boldsymbol{\Psi}^{n-m}(\boldsymbol{x}),\, C_{j,n}\; \boldsymbol{\Phi}^m \otimes_s \boldsymbol{\Phi}^{n-m}(\boldsymbol{x})\rangle \\ &\qquad = \langle \boldsymbol{\Psi}^m,\, C_{j,m}\, \boldsymbol{\Phi}^m\rangle \langle \boldsymbol{\Psi}^{n-m}, \boldsymbol{\Phi}^{n-m}\rangle, \quad 0 \leq j \leq m\,.\end{aligned}$$

Comparing this equality with the result obtained in step (i), it follows that

$$A \upharpoonright \mathcal{F}_m = C_{0,m} + C_{1,m} + \cdots + C_{m,m} = \kappa_{m,n}(K_n)\,,$$

completing the proof. $\square$

In view of the preceding lemma the following picture of the algebra $\mathfrak{A}$ emerges: each $A \in \mathfrak{A}$ determines a bounded sequence $\{A \upharpoonright \mathcal{F}_n = K_n(A) \in \mathfrak{K}_n\}_{n\in\mathbb{N}_0}$ that satisfies the coherence condition $\kappa_n(K_n(A)) = K_{n-1}(A)$, $n \in \mathbb{N}_0$. It is thus a special element of the inverse system of C*-algebras $(\mathfrak{K}_n, \kappa_n)_{n\in\mathbb{N}_0}$, cf. [19]. To characterize such elements we introduce the following definition.

Definition: The linear space of bounded sequences $\boldsymbol{K} \doteq \{K_n \in \mathfrak{K}_n\}_{n\in\mathbb{N}_0}$ satisfying the coherence condition $\kappa_n(K_n) = K_{n-1}$, $n \in \mathbb{N}_0$, is denoted by $\boldsymbol{\mathfrak{K}}$. It is a Banach space with regard to the norm $\|\boldsymbol{K}\|_\infty \doteq \sup_n \|K_n\|_n$ and coincides, as a set, with the closure of its bounded subsets in the locally convex topology given by the seminorms $\|\cdot\|_n$, $n \in \mathbb{N}_0$. By some abuse of terminology, $\boldsymbol{\mathfrak{K}}$ is called the inverse limit of the inverse system $(\mathfrak{K}_n, \kappa_n)_{n\in\mathbb{N}_0}$.

It follows from the preceding discussion that $\boldsymbol{\mathfrak{K}}$ is actually an algebra, where products and the star operation are componentwise defined. In order to verify this we proceed to a slight extension of the algebra $\mathfrak{A}$.

Definition: The algebra $\overline{\mathfrak{A}}$ is defined by completion of the bounded subsets of $\mathfrak{A}$ with regard to the locally convex topology induced by the seminorms $\|\cdot\|_n$, $n \in \mathbb{N}_0$. Its elements correspond to those (sequential) limits of $\mathfrak{A}$ in the strong operator topology on $\mathcal{F}$ whose restrictions to each subspace $\mathcal{F}_n$ coincide with some element of $\mathfrak{A}$, $n \in \mathbb{N}_0$.

Any element $A \in \overline{\mathfrak{A}}$ defines a coherent sequence $\boldsymbol{K}(A) = \{A \upharpoonright \mathcal{F}_n\}_{n\in\mathbb{N}_0}$. But the converse is also true: given a sequence $\boldsymbol{K} = \{K_n \in \mathfrak{K}_n\}_{n\in\mathbb{N}_0} \in \boldsymbol{\mathfrak{K}}$, it follows from Lemma 3.3 that for each $M \in \mathbb{N}$ there is some operator $A_M \in \mathfrak{A}$ such that $A_M \upharpoonright \mathcal{F}_m = K_m$, $0 \le m \le M$. The limit of the sequence $\{A_M\}_{M\in\mathbb{N}}$ on $\mathcal{F}$, which is denoted by A, satisfies $\{A \upharpoonright \mathcal{F}_n = K_n\}_{n\in\mathbb{N}_0}$, hence it reproduces the given sequence $\boldsymbol{K}$. Since $\overline{\mathfrak{A}}$ is a C*-algebra with regard to the norm $\|\cdot\|_\infty$, it follows that $\boldsymbol{\mathfrak{K}}$ is a C*-algebra as well. In fact, given sequences $\boldsymbol{K}_1, \boldsymbol{K}_2 \in \boldsymbol{\mathfrak{K}}$ and corresponding operators $A_1, A_2 \in \overline{\mathfrak{A}}$ one has

$$K_{1n}K_{2n} = A_1A_2 \upharpoonright \mathcal{F}_n\,, \quad K_{1n}^* = A_1^* \upharpoonright \mathcal{F}_n\,, \quad n \in \mathbb{N}_0\,,$$

and, as explained, A_1A_2, A_1^* determine elements of $\boldsymbol{\mathfrak{K}}$.

We summarize the preceding results in the following theorem.

Theorem 3.5. *The map*

$$A \mapsto \{A \upharpoonright \mathcal{F}_n \in \mathfrak{K}_n\}_{n\in\mathbb{N}_0}\,, \quad A \in \overline{\mathfrak{A}}\,,$$

establishes an isomorphism between the C-algebra $\overline{\mathfrak{A}}$ and the inverse limit $\boldsymbol{\mathfrak{K}}$ of the inverse system of approximately finite dimensional C*-algebras $(\mathfrak{K}_n, \kappa_n)_{n\in\mathbb{N}_0}$.*

4 Dynamics of observables

With this information about the structure of the algebra $\overline{\mathfrak{A}}$, we can turn now to the discussion of dynamics. We will take advantage of the fact that the elements of the algebra $\overline{\mathfrak{A}}$ and the Hamiltonians under consideration commute with the particle number operator. So we can fix $n \in \mathbb{N}$ and restrict these operators to the subspaces $\mathcal{F}_n \subset \mathcal{F}$. We recall that $\overline{\mathfrak{A}} \upharpoonright \mathcal{F}_n = \mathfrak{K}_n$, the algebra generated by symmetrized compact operators and the unit operator. The restriction of the Hamiltonian $H \upharpoonright \mathcal{F}_n = H_n$ can be expressed in terms of position and momentum operators, cf. equation (2.1).

It will be convenient in the analysis to extend the restricted Hamiltonian H_n to the unsymmetric space $\mathcal{H}_n$ by maintaining the definition of the underlying position and momentum operators. We also extend the symmetric algebra $\mathfrak{K}_n$ on $\mathcal{F}_n$ and embed it into the algebra $\mathfrak{K}(\mathbb{I}_n)$ generated by all unsymmetrized compact operators acting on the tensor factors of $\mathcal{H}_n$. In more detail, given any ordered subset $\mathbb{I}_m \doteq \{i_1, \dots, i_m\} \subset \mathbb{I}_n$ of $m \leq n$ elements, we consider on $\mathcal{H}_n$ the unique C*-tensor product

$$\mathfrak{C}_n(\mathbb{I}_m) \doteq \mathfrak{C}(i_1) \otimes \cdots \otimes \mathfrak{C}(i_m)$$

generated by the commuting algebras of compact operators acting on the corresponding tensor factors of $\mathcal{H}_n$. Here we have omitted tensor factors of 1 acting on the remaining components. The unital C*-algebra $\mathfrak{K}(\mathbb{I}_n)$ is then defined as the linear span of the algebras $\mathfrak{C}_n(\mathbb{I}_m)$ for all $\mathbb{I}_m \subset \mathbb{I}_n$ and $0 \leq m \leq n$, where we put $\mathbb{I}_0 = \emptyset$ and $\mathfrak{C}_n(\mathbb{I}_0) \doteq \mathbb{C}\,\mathbf{1}$. The symmetric algebra $\mathfrak{K}_n$ is identified with the subalgebra of $\mathfrak{K}(\mathbb{I}_n)$ consisting of all operators which commute with the projection $\overline{U}_n(\Sigma_n)$.

Making use of arguments established in [8] for distinguishable particles oscillating about lattice points, we will show that the algebra $\mathfrak{K}(\mathbb{I}_n) \subset \mathcal{B}(\mathcal{H}_n)$ is stable under the adjoint action of the unitaries e^{itH_n} on $\mathcal{B}(\mathcal{H}_n)$, determined by the Hamiltonian H_n. This action is denoted by $\alpha_n(t) \doteq \mathrm{Ad}\, e^{itH_n}$, $t \in \mathbb{R}$. We recall these arguments here in some detail.

For potential $V = 0$ one obtains the non-interacting Hamiltonian $H_{0\,n}$. The corresponding adjoint action $\alpha_n^{(0)}(t) \doteq \mathrm{Ad}\, e^{itH_{0n}}$ leaves the subalgebra $\mathfrak{K}(\mathbb{I}_n) \subset \mathcal{B}(\mathcal{H}_n)$ invariant. This is apparent since the unitaries $e^{itH_{0n}}$ do not mix the tensor factors of $\mathcal{H}_n$ and the adjoint action of unitary operators maps compact operators onto compact operators, $t \in \mathbb{R}$. Morover, since the function $t \mapsto e^{itH_{0n}}$ is continuous in the strong operator topology, it is also clear that $t \mapsto \alpha_n^{(0)}(t)$ acts pointwise norm-continuously on $\mathfrak{K}(\mathbb{I}_n)$.

Next, we consider the familiar Dyson cocyles $\Gamma_n(t) \doteq e^{itH_n} e^{-itH_{0\,n}}$, $t \in \mathbb{R}$. They define automorphisms $\gamma_n(t) : \mathcal{B}(\mathcal{H}_n) \to \mathcal{B}(\mathcal{H}_n)$ given by

$$\gamma_n(t) \doteq \mathrm{Ad}\,\Gamma_n(t)\,, \quad t \in \mathbb{R}\,. \tag{4.1}$$

We must show that their restrictions to $\mathfrak{K}(\mathbb{I}_n) \subset \mathcal{B}(\mathcal{H}_n)$ map this subalgebra onto itself. For this implies by the preceding remarks that the algebra is stable under the automorphic action $\alpha_n(t) = \gamma_n(t) \circ \alpha_n^{(0)}(t)$, $t \in \mathbb{R}$, of the given dynamics. As a matter of fact, it suffices to establish the inclusion $\gamma_n(t)(\mathfrak{K}(\mathbb{I}_n)) \subset \mathfrak{K}(\mathbb{I}_n)$ since according to the preceding remarks, this implies $\alpha_n(t)(\mathfrak{K}(\mathbb{I}_n)) \subset \mathfrak{K}(\mathbb{I}_n)$ and one has $\alpha_n(t)^{-1} = \alpha_n(-t)$, $t \in \mathbb{R}$.

We pick any $C \in \mathfrak{K}(\mathbb{I}_n)$ and consider the familiar Dyson expansion

$$\gamma_n(t)(C) = C + \sum_{l=1}^{\infty} i^l \int_0^t ds_l \int_0^{s_l} ds_{l-1} \cdots \int_0^{s_2} ds_1 [\dots [C, \boldsymbol{V}_n(s_1)] \dots, \boldsymbol{V}_n(s_l)]\,. \tag{4.2}$$

Here we have introduced the short hand notation $\boldsymbol{V}_n \doteq \sum_{j \neq k} V(\boldsymbol{Q}_j - \boldsymbol{Q}_k)$ and put $\boldsymbol{V}_n(s) \doteq \alpha_n^{(0)}(s)(\boldsymbol{V}_n)$, $s \in \mathbb{R}$. The integrals are defined in the strong operator topology on $\mathcal{H}_n$ and the sum (4.1) converges absolutely in norm, uniformly on compact subsets of $t \in \mathbb{R}$, because $\boldsymbol{V}_n$ is a bounded operator. Adopting arguments from [8], we obtain the following result for the derivations appearing in the multiple commutators in this expansion.

Lemma 4.1. *Let $\boldsymbol{\delta}_n(s)$, $s \in \mathbb{R}$, be the bounded derivations on $\mathcal{B}(\mathcal{H}_n)$ given by*

$$\boldsymbol{\delta}_n(s)(B) \doteq \int_0^s du\,[B, \boldsymbol{V}_n(u)]\,, \quad B \in \mathcal{B}(\mathcal{H}_n)\,.$$

The subalgebra $\mathfrak{K}(\mathbb{I}_n) \subset \mathcal{B}(\mathcal{H}_n)$ is stable under their action, $\boldsymbol{\delta}_n(s)(\mathfrak{K}(\mathbb{I}_n)) \subset \mathfrak{K}(\mathbb{I}_n)$. Moreover, the function $s \mapsto \boldsymbol{\delta}_n(s)$ acts pointwise norm-continuously on $\mathfrak{K}(\mathbb{I}_n)$ and is bounded on compact subsets of $s \in \mathbb{R}$.

Remark: The bounded operators $\int_0^s du \boldsymbol{V}_n(u)$ are not contained in $\mathfrak{K}(\mathbb{I}_n)$. It is essential for this result that the C*-algebra $\mathfrak{K}(\mathbb{I}_n)$ is not simple (*i.e.* has ideals). Outer bounded derivations can and do exist in such cases.

Proof. Since $\boldsymbol{V}_n(u)$, $u \in \mathbb{R}$, is a finite sum of operators, it suffices to establish the statement for its summands which are given by, $j, k \in \mathbb{I}_n$ and $j \neq k$,

$$V_{jk}(u) \doteq \alpha_n^{(0)}(u)(V(\boldsymbol{Q}_j - \boldsymbol{Q}_k)) = V(\boldsymbol{Q}_j - \boldsymbol{Q}_k + 2u(\boldsymbol{P}_j - \boldsymbol{P}_k))\,, \quad u \in \mathbb{R}\,.$$

These operators are elements of the C*-algebra $\mathfrak{R}_n(j \frown k)$, which is generated by all continuous functions vanishing at infinity of $\big(\boldsymbol{a}\,(\boldsymbol{Q}_j - \boldsymbol{Q}_k) + \boldsymbol{b}\,(\boldsymbol{P}_j - \boldsymbol{P}_k)\big)$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^s$.

As a matter of fact, this algebra coincides with the resolvent algebra generated by these linear combinations of canonically conjugate operators and it is faithfully represented on $\mathcal{H}_n$, cf. [9, Thm. 4.10]. The algebra is stable under the action of the automorphisms $\alpha_n^{(0)}(u)$, $u \in \mathbb{R}$, and it contains a unique minimal ideal $\mathfrak{C}_n(j \frown k) \subset \mathfrak{R}_n(j \frown k)$ which is isomorphic to the algebra of compact operators on $L^2(\mathbb{R}^s)$, cf. [9, Thm. 5.4].

The first and vital step consists of the proof that the integrals $\int_0^s du\, V_{jk}(u)$ (defined in the strong operator topology) are elements of the compact ideal $\mathfrak{C}_n(j \frown k)$. A similar result was established in [8, Lem. 2.1] for the case, where the present non-interacting Hamiltonian is replaced by the Hamiltonian of an isotropic harmonic oscillator. That Hamiltonian also induces an automorphic action of time translations on the algebra $\mathfrak{R}_n(j \frown k)$. The argument given in [8] can be applied in the present case without major modifications and is put into the appendix.

Knowing that the integrals $\int_0^s du\, V_{jk}(u)$, $s \in \mathbb{R}$, belong to the compact ideal $\mathfrak{C}_n(j \frown k)$, we can proceed and apply the same arguments as in [8, Lem. 2.2]: we pick indices $\mathbb{I}_m \subset \mathbb{I}_n$ and consider the corresponding algebra $\mathfrak{C}_n(\mathbb{I}_m) \subset \mathfrak{K}(\mathbb{I}_n)$. Let $C \in \mathfrak{C}_n(\mathbb{I}_m)$, there then appear four different possibilities for the action of the derivation $\delta_{jk}(s)$ on C, given by the commutator with $\int_0^s du\, V_{jk}(u)$.

(i) If both indices $j, k \notin \mathbb{I}_m$, then $V_{jk}(u)$ commutes with C for any $u \in \mathbb{R}$, and consequently $\delta_{jk}(s)(C) = 0$ for $s \in \mathbb{R}$.

(ii) If both indices $j, k \in \mathbb{I}_m$, then $\mathfrak{R}_n(j \frown k)$, hence $\mathfrak{C}_n(j \frown k)$, lies in the multiplier algebra of the algebra $\mathfrak{C}_n(\mathbb{I}_m)$. Thus $\delta_{jk}(s)(C) \in \mathfrak{C}_n(\mathbb{I}_m)$, $s \in \mathbb{R}$.

(iii) If $k \in \mathbb{I}_m$, but $j \notin \mathbb{I}_m$, then

$$\delta_{jk}(s)(C) \in [\mathfrak{C}_n(j \frown k),\, \mathfrak{C}_n(\mathbb{I}_m)]\,.$$

As has been shown in [8, Lem. 2.2], these commutators are elements of the algebra $\mathfrak{C}_n(\mathbb{I}_{m+1})$, where $\mathbb{I}_{m+1} \doteq \mathbb{I}_m \overset{o}{\cup} j$ and the union symbol $\overset{o}{\cup}$ indicates that the index j is to be inserted at its proper place within the ordered set $\mathbb{I}_{m+1}$. Thus $\delta_{jk}(s)(C) \in \mathfrak{C}_n(\mathbb{I}_{m+1})$, $s \in \mathbb{R}$.

(iv) Finally, if $j \in \mathbb{I}_m$, but $k \notin \mathbb{I}_m$, then the same argument as in the preceding step shows that $\delta_{jk}(s)(C) \in \mathfrak{C}_n(\mathbb{I}_{m+1})$, $s \in \mathbb{R}$, where now $\mathbb{I}_{m+1} \doteq \mathbb{I}_m \overset{o}{\cup} k$ and k has to be inserted at its proper place.

Since the algebra $\mathfrak{K}(\mathbb{I}_n)$ is equal to the linear span of the algebras $\mathfrak{C}_n(\mathbb{I}_m)$ for arbitrary ordered index sets $\mathbb{I}_m \subset \mathbb{I}_n$ and $0 \leq m \leq n$, we conclude that $\boldsymbol{\delta}_n(s)(\mathfrak{K}(\mathbb{I}_n)) \subset \mathfrak{K}(\mathbb{I}_n)$, $s \in \mathbb{R}$. The remaining statements about the function $s \mapsto \boldsymbol{\delta}_n(s)$ are clearly valid for their action on $\mathcal{B}(\mathcal{H}_n)$, hence *a fortiori* for their restriction to its subalgebra $\mathfrak{K}(\mathbb{I}_n)$. This completes the proof of the lemma. $\square$

With the help of this lemma we can show now by induction, similarly to the argument in [8, Lem. 3.1], that each summand in the above Dyson expansion is contained in $\mathfrak{K}(\mathbb{I}_n)$. To this end we express the respective l-fold integrals in terms of the derivations defined in the lemma. This gives, $C \in \mathfrak{K}(\mathbb{I}_n)$,

$$D_l(t)(C) \doteq \int_0^t ds_l \int_0^{s_l} ds_{l-1} \cdots \int_0^{s_2} ds_1 \, \boldsymbol{\delta}_n(s_l) \circ \boldsymbol{\delta}_n(s_{l-1}) \circ \cdots \circ \boldsymbol{\delta}_n(s_1) \, (C) \, .$$

For $l = 1$ it follows from the preceding lemma that $D_1(t)(C) \in \mathfrak{K}(\mathbb{I}_n)$, $t \in \mathbb{R}$. Moreover, the function $t \mapsto D_1(t)(C)$ is continuous in norm and bounded on compact subsets of $\mathbb{R}$. The induction hypothesis consists of the assertion that $D_l(t)(C) \in \mathfrak{K}(\mathbb{I}_n)$ and $t \mapsto D_l(t)(C)$ is norm continuous and locally bounded for given $l \in \mathbb{N}$. For the induction step from l to $l+1$ we notice that

$$D_{l+1}(t)(C) = \int_0^t ds \, \boldsymbol{\delta}_n(s)(D_l(s)(C)) = \lim_{M \to \infty} \sum_{m=0}^{M-1} \int_{mt/M}^{(m+1)t/M} ds \, \boldsymbol{\delta}_n(s)(D_l(s_m)(C)) \, ,$$

where all integrals are defined in the strong operator topology. In the second equality, we made use of the induction hypothesis, according to which $s \mapsto D_l(s)(C) \in \mathfrak{K}(\mathbb{I}_n)$ is norm continuous. This function can therefore be approximated by its values at the boundary points of the chosen partition of the interval $[0, t]$. Moreover, according to Lemma 4.1, the function $s \mapsto \boldsymbol{\delta}_n(s)$ is bounded on compact sets of $\mathbb{R}$. Hence the sum of integrals on the right hand side of the second equality converges in norm to the expression on its left hand side.

Finally, according to Lemma 4.1, the function $s \mapsto \boldsymbol{\delta}_n(s)$ is pointwise norm continuous on $\mathfrak{K}(\mathbb{I}_n)$ and maps this algebra into itself. Hence each term in the sum on the right hand side of the second equality is an element of $\mathfrak{K}(\mathbb{I}_n)$. Since the sum is norm convergent, it follows that $D_{l+1}(t)(C) \in \mathfrak{K}(\mathbb{I}_n)$. It is also apparent that the function $t \mapsto D_{l+1}(t)(C)$ is norm continuous and locally bounded. This completes the induction. In view of the absolute convergence of the Dyson series in the norm topology, we have thus established the following fact.

Lemma 4.2. *Let $n \in \mathbb{N}$. The Dyson automorphisms $\gamma_n(t)$, $t \in \mathbb{R}$, defined in relation* (4.1)*, map the subalgebra $\mathfrak{K}(\mathbb{I}_n) \subset \mathcal{B}(\mathcal{H}_n)$ onto itself, i.e. they are automorphisms of this algebra. Moreover, the function $t \mapsto \gamma_n(t)$ is pointwise norm-continuous on this algebra.*

It is now easy to show that the symmetric subalgebra $\mathfrak{K}_n \subset \mathfrak{K}(\mathbb{I}_n)$ of interest here is stable under the interacting dynamics. At this point we make use of the fact that the

Hamiltonians H_n, which were extended from equation (2.1) to $\mathcal{H}_n$, commute with the projection $\overline{U}_n(\Sigma_n)$ onto $\mathcal{F}_n$. Hence the Dyson automorphisms $\gamma_n(t)$ of $\mathfrak{K}(\mathbb{I}_n)$ commute with the operation of symmetrization, $t \in \mathbb{R}$. It therefore follows from the preceding lemma that $\gamma_n(t)(\mathfrak{K}_n) = \mathfrak{K}_n$, $t \in \mathbb{R}$. Bearing in mind the properties of the non-interacting time evolution, this implies

$$\alpha_n(t)(\mathfrak{K}_n) = \gamma_n(t) \circ \alpha_n^{(0)}(t)(\mathfrak{K}_n) = \mathfrak{K}_n \,, \quad t \in \mathbb{R} \,.$$

As has been explained, the action of $t \mapsto \alpha_n^{(0)}(t)$ on $\mathfrak{K}_n$ is pointwise norm continuous, hence by the preceding results we arrive at the following proposition.

Proposition 4.3. *Let $n \in \mathbb{N}$. Given any dynamics, fixed by a Hamiltonian H_n as in relation* (2.1)*, one has $\alpha_n(t)(\mathfrak{K}_n) = \mathfrak{K}_n$. Moreover, the function $t \mapsto \alpha_n(t)$ is pointwise norm continuous on $\mathfrak{K}_n$, $t \in \mathbb{R}$.*

Having determined the properties of the action of the dynamics $\alpha_n(t)$, $t \in \mathbb{R}$, on the restricted algebras $\overline{\mathfrak{A}} \upharpoonright \mathcal{F}_n = \mathfrak{K}_n$, $n \in \mathbb{N}_0$, we turn now to the full algebra. To this end we consider the unitary operators e^{itH}, $t \in \mathbb{R}$, on Fock space $\mathcal{F}$, fixed by the Hamiltonian H in equation (1.1). Their adjoint action on $\mathcal{B}(\mathcal{F})$ is denoted by $\boldsymbol{\alpha}(t) \doteq \mathrm{Ad}\, e^{itH}$, $t \in \mathbb{R}$. Recalling that H commutes with the particle number operator, we have $e^{itH} \upharpoonright \mathcal{F}_n = e^{itH_n}$ with H_n as in equation (2.1). Moreover, $\boldsymbol{\alpha}(t) \upharpoonright \mathcal{B}(\mathcal{F}_n) = \alpha_n(t)$, $t \in \mathbb{R}$, where $\mathcal{B}(\mathcal{F}_n)$ is embedded into $\mathcal{B}(\mathcal{F})$ by putting $\mathcal{B}(\mathcal{F}_n) \upharpoonright \mathcal{F}_m = 0$ for $m \neq n$.

Given $A \in \overline{\mathfrak{A}}$, it follows from Theorem 3.5 that $A \upharpoonright \mathcal{F}_n = K_n \in \mathcal{K}_n$ and according to Proposition 4.3 we have $\alpha_n(t)(K_n) \in \mathfrak{K}_n$ for $t \in \mathbb{R}$ and any $n \in \mathbb{N}_0$. Thus the sequence $\{\boldsymbol{\alpha}(t)(A) \upharpoonright \mathcal{F}_n \in \mathfrak{K}_n\}_{n \in \mathbb{N}_0}$ defines for each $t \in \mathbb{R}$ some element of the directed system $(\mathfrak{K}_n, \epsilon_n)_{n \in \mathbb{N}_0}$. In order to prove that this sequence corresponds to some element of $\overline{\mathfrak{A}}$, we have to show that it defines an element of the inverse limit $\boldsymbol{\mathfrak{K}}$ of $(\mathfrak{K}_n, \kappa_n)_{n \in \mathbb{N}_0}$. For the proof of this we make use of the subsequent lemma. The desired result then follows from Theorem 3.5.

Lemma 4.4. *Let $\boldsymbol{\alpha}(t)$, $t \in \mathbb{R}$, be the one-parameter group of automorphisms on $\mathcal{B}(\mathcal{F})$, fixed by a Hamiltonian as in relation* (1.1)*, and let $\alpha_n(t)$, $t \in \mathbb{R}$, be its restrictions to $\mathfrak{K}_n \subset \mathcal{B}(\mathcal{F}_n)$, $n \in \mathbb{N}_0$. Then*

$$\alpha_{n-1}(t) \circ \kappa_n = \kappa_n \circ \alpha_n(t) \,, \quad n \in \mathbb{N} \,.$$

Thus $\boldsymbol{\alpha}(t)$, $t \in \mathbb{R}$, induces an automorphic action on the inverse system $(\mathfrak{K}_n, \kappa_n)_{n \in \mathbb{N}}$ which preserves its inverse limit $\boldsymbol{\mathfrak{K}}$.

Proof. Given $n \in \mathbb{N}$, we consider the restriction of the non-interacting dynamics $\alpha_n^{(0)}(t)$, $t \in \mathbb{R}$, to the unsymmetrized algebra $\mathfrak{K}(\mathbb{I}_n)$. As was explained, this dynamics leaves each subalgebra $\mathfrak{C}(\mathbb{I}_m) \subset \mathfrak{K}(\mathbb{I}_n)$ invariant. This implies after symmetrization

$$\alpha_n^{(0)}(t) \upharpoonright (\mathfrak{C}_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m}) = (\alpha_m^{(0)}(t) \upharpoonright \mathfrak{C}_m) \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m}, \quad 0 \leq m \leq n\,.$$

The statement then follows for the non-interacting dynamics from the definition of the inverse map κ_n on $\mathfrak{K}_n$, $n \in \mathbb{N}_0$.

In view of this fact it suffices to establish the modified statement of the lemma, where the automorphisms $\alpha_\bullet(t)$ are replaced by the Dyson cocycles $\gamma_\bullet(t)$, defined in relation (4.1). As a matter of fact, relying on the norm convergence of the resulting Dyson expansion, it suffices to restrict attention to the underlying derivations $\boldsymbol{\delta}_\bullet(s)$, defined in Lemma 4.1, and to show that their restrictions to $\mathfrak{K}_n$ satisfy for all $s \in \mathbb{R}$

$$\boldsymbol{\delta}_{n-1}(s) \circ \kappa_n = \kappa_n \circ \boldsymbol{\delta}_n(s)\,, \quad n \in \mathbb{N}\,,$$

where $\boldsymbol{\delta}_0(s) \doteq 0$. The statement then follows.

We proceed now as in the proof of Lemma 4.1 and decompose the derivations $\boldsymbol{\delta}_n(s)$ into their building blocks, $\boldsymbol{\delta}_n(s) = \sum_{j \neq k} \delta_{jk}(s)$, where $j, k \in \mathbb{I}_n$. We also use the notation $\boldsymbol{\delta}_{\mathbb{I}}(s)$ if the summation extends over $j, k \in \mathbb{I}$, where $\mathbb{I} \subset \mathbb{I}_n$ is any ordered index set.

Let $1 \leq m \leq n$ and let $\mathfrak{C}_n(\mathbb{I}_m) \subset \mathfrak{K}(\mathbb{I}_n)$ be the subalgebra corresponding to the index set $\mathbb{I}_m = \{1, \dots, m\} \subset \mathbb{I}_n$. Other choices of $\mathbb{I}_m$ are treated similarly. Bearing in mind the specific properties of the action of the derivations $\delta_{jk}(s)$ on the algebras $\mathfrak{C}_n(\mathbb{I}_m)$ established in Lemma 4.1, we obtain for the unsymmetrized tensor products

$$\begin{aligned}
&\boldsymbol{\delta}_n(s) \upharpoonright (\mathfrak{C}_n(\mathbb{I}_m) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-m}) = (\boldsymbol{\delta}_{\mathbb{I}_m}(s) \upharpoonright \mathfrak{C}_n(\mathbb{I}_m)) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-m} \\
&+ \sum_{k \in \mathbb{I}_n \backslash \mathbb{I}_m} (\boldsymbol{\delta}_{\mathbb{I}_m \overset{o}{\cup} k}(s) - \boldsymbol{\delta}_{\mathbb{I}_m}(s) \otimes \iota_k) \upharpoonright (\mathfrak{C}_n(\mathbb{I}_m) \otimes \underbrace{1 \otimes \cdots \otimes \overset{k}{1} \otimes \cdots \otimes 1}_{n-m})\,.
\end{aligned}$$

Here $\mathbb{I}_m \overset{o}{\cup} k$ denotes the ordered index obtained by adding to $\mathbb{I}_m$ the element $k \in \mathbb{I}_n \backslash \mathbb{I}_m$. The derivations in the sum on the second line act on factors located at $\mathbb{I}_m \overset{o}{\cup} k$, where ι_k denotes the identity map at k. The sum vanishes if $m = n$. Analogous relations hold for any other choice of index set $\mathbb{I}_m \subset \mathbb{I}_n$. Taking into account that $\boldsymbol{\delta}_n(s)$ commutes with the operation of symmetrization, we can proceed from the preceding relation to the corresponding version on the symmetrized subalgebras $\mathfrak{C}_{m,n} \subset \mathfrak{K}(\mathbb{I}_n)$ by taking averages over the symmetric group Σ_n.

Now if $m = n$, then the right hand side of the symmetrized version of the above equality is contained in $\mathfrak{C}_n$, cf. the proof of Lemma 4.1. Hence it is mapped to 0 by κ_n. If $m = n-1$, the second term on the right hand side of the symmetrized equality is contained in $\mathfrak{C}_n$ and mapped to 0 by κ_n, as well. The first term is of the form $(\boldsymbol{\delta}_{n-1}(s) \upharpoonright \mathfrak{C}_{n-1}) \otimes_s 1$ and thus mapped by κ_n to $\boldsymbol{\delta}_{n-1}(s) \upharpoonright \mathfrak{C}_{n-1}$, in accordance with the statement. If $m < n-1$, each term of the above equality contains uninvolved (spectator) tensor factors 1, hence κ_n maps its symmetrized version into an equality of the same form, where on the right hand side n is to be replaced by $n-1$. Reading the resulting equality from right to left, it follows that

$$\kappa_n \circ \boldsymbol{\delta}_n(s) \upharpoonright (\mathfrak{C}_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m}) = \boldsymbol{\delta}_{n-1}(s) \upharpoonright (\mathfrak{C}_m \otimes_s \underbrace{1 \otimes_s \cdots \otimes_s 1}_{n-m-1}), \quad m < n-1\,,$$

completing the proof of the lemma. □

We have now the necessary information for the proof that any Hamiltonian of the form given in equation (1.1) induces an automorphic action of dynamics on the algebra of observables $\overline{\mathfrak{A}}$.

Theorem 4.5. *Let $\boldsymbol{\alpha}(t)$, $t \in \mathbb{R}$, be the group of automorphisms of $\mathcal{B}(\mathcal{F})$ fixed by a Hamiltonian of the form given in equation* (1.1)*. These automorphisms map the observable algebra $\overline{\mathfrak{A}} \subset \mathcal{B}(\mathcal{F})$ onto itself and they act pointwise continuously on this algebra with regard to the locally convex topology induced by the seminorms $\| \cdot \|_n$, $n \in \mathbb{N}_0$. Moreover, there is a (in this topology) dense subalgebra $\overline{\mathfrak{A}}_\alpha \subset \overline{\mathfrak{A}}$ on which the automorphisms act pointwise norm continuously with regard to the C^*-norm on $\overline{\mathfrak{A}}$. Thus $(\overline{\mathfrak{A}}_\alpha, \alpha)$ is a C^*-dynamical system.*

Proof. It was shown in Lemma 4.4 that $\boldsymbol{\alpha}(t)$, $t \in \mathbb{R}$, defines a group of automorphisms of the inverse limit $\boldsymbol{\mathfrak{K}}$ of $(\mathfrak{K}_n, \kappa_n)_{n\in\mathbb{N}}$. The algebra $\boldsymbol{\mathfrak{K}}$ in turn is isomorphic to $\overline{\mathfrak{A}}$ according to Theorem 3.5, proving the first part of the statement. The stated continuity properties of theses automorphisms with regard to the locally convex topology then follow from Proposition 4.3. These continuity properties imply in particular that the mollified operators $\int ds\, f(s)\boldsymbol{\alpha}(s)(A)$, where $f \in L^1(\mathbb{R})$, $A \in \overline{\mathfrak{A}}$, generate a unital subalgebra $\overline{\mathfrak{A}}_\alpha \subset \overline{\mathfrak{A}}$ which is dense in $\overline{\mathfrak{A}}$ in the locally convex topology. Note that the restrictions of the integrals to the subspaces $\mathcal{F}_n \subset \mathcal{F}$, $n \in \mathbb{N}_0$, are defined in the norm topology. The stronger continuity properties of the elements of $\overline{\mathfrak{A}}_\alpha$ with regard to the action of the automorphisms then follow from the estimate $\| \int ds\, f(s)\boldsymbol{\alpha}(s)(A)\|_\infty \leq \int ds\, |f(s)|\, \|A\|_\infty$ and the well known continuity properties of the elements of $L^1(\mathbb{R})$ with regard to translations. □

We conclude this section by discussing locality properties of the observables, which are a distinctive feature of field theory, having no counter part in the particle picture. The resolvent algebra $\mathfrak{R}$ is, by construction, the C*-inductive limit of the net of its subalgebras $\mathfrak{R}(\boldsymbol{O})$ based on the open, bounded regions $\boldsymbol{O} \subset \mathbb{R}^s$. The algebras $\mathfrak{R}(\boldsymbol{O})$ are generated by the subsets of resolvents $R(\lambda, f)$, where $\lambda \in \mathbb{R}\backslash\{0\}$ and $f \in \mathcal{D}(\boldsymbol{O})$, the space of test functions having support in $\boldsymbol{O}$. Since the algebras $\mathfrak{R}(\boldsymbol{O})$ are stable under gauge transformations, we can proceed to their gauge invariant subalgebras $\mathfrak{A}(\boldsymbol{O}) \subset \mathfrak{R}(\boldsymbol{O})$, $\boldsymbol{O} \subset \mathbb{R}^s$, and the algebra of all gauge invariant observables $\mathfrak{A}$ is the C*-inductive limit of these subalgebras. It is an immediate consequence of the canonical commutation relations that elements of algebras assigned to disjoint regions in $\mathbb{R}^s$ commute with each other, thereby implementing the principle of locality (statistical independence of spatially separated observables) at the kinematical level.

These locality properties carry over to the algebras $\overline{\mathfrak{A}}(\boldsymbol{O})$, $\boldsymbol{O} \subset \mathbb{R}^s$, obtained by completing each $\mathfrak{A}(\boldsymbol{O})$ in the locally convex topology induced by the seminorms $\| \cdot \|_n$, $n \in \mathbb{N}_0$. Moreover, the algebra of observables $\overline{\mathfrak{A}}$ is the limit of its local subalgebras relative to the underlying locally convex topology. In fact, given any $A \in \overline{\mathfrak{A}}$, there exists a sequence of operators $\{A_N \in \mathfrak{A}(\boldsymbol{O}_N)\}_{N\in\mathbb{N}}$, which are localized in bounded regions $\boldsymbol{O}_N$, such that $\lim_{N\rightarrow\infty} \|A_N - A\|_n = 0$, $n \in \mathbb{N}_0$. This sequence is obtained by approximating the operators $A \restriction \mathcal{F}_n$ through local operators $A_N \restriction \mathcal{F}_n$, which is possible according to Lemma 3.3 and the definition of $\overline{\mathfrak{A}}$. The approximations have to be progressively improved for increasing n and, taking into account that the semi-norms $\| \cdot \|_n$ are increasing with $n \in \mathbb{N}_0$, one thereby obtains the desired approximating sequence.

Whereas the non-relativistic dynamics does not preserve the kinematical local structure of the observables, our results imply that there remain some quasilocal properties of the time translated observables in the following sense. Let $\boldsymbol{\alpha}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^s$, be the automorphism group of spatial translations on $\mathfrak{A}$ which acts on the generating resolvents according to $\boldsymbol{\alpha}(\boldsymbol{x})(R(\lambda, f)) \doteq R(\lambda, f_{\boldsymbol{x}})$, where $f_{\boldsymbol{x}}$ denotes the test function f, translated by $\boldsymbol{x}$. These automorphisms commute with the time translations $\boldsymbol{\alpha}(t)$, $t \in \mathbb{R}$, and can be extended to the algebra $\overline{\mathfrak{A}}$. Hence, putting $\boldsymbol{\alpha}(t, \boldsymbol{x}) \doteq \boldsymbol{\alpha}(t) \circ \boldsymbol{\alpha}(\boldsymbol{x})$ and taking into account the preceding remarks, we obtain the following corollary of Theorem 4.5.

Corollary 4.6. *Let $A, B \in \overline{\mathfrak{A}}$ and let $t \in \mathbb{R}$. Then*

$$\lim_{\boldsymbol{x}\rightarrow\infty} \| [\boldsymbol{\alpha}(t, \boldsymbol{x})(A), B] \|_n = 0 \, , \quad n \in \mathbb{N}_0 \, .$$

So the time translated observables are still quasilocal in this sense. It is an interesting question whether for given interaction potential one can establish more specific bounds for these commutators, *e.g.* of Lieb-Robinson type [16], which would indicate limitations on the speed of propagation of causal influences.

5 States

Having settled the framework, we turn now to some applications in many body theory. As already mentioned in the introduction, it is the primary purpose of this part of our article to convey ideas, functional analytic details are largely omitted.

5.1 Ground states

The pair potentials V considered here may lead to bound states whose energy lies below the energy of the vacuum state Ω, which is put equal to 0 by the definition of the Hamiltonians H in equation (1.1). This defect can be resolved by renormalizing the Hamiltonians, which can be accomplished without affecting the dynamics of the observables.

Let H be given and let $-E(n)$ be the infimum of the spectrum of $H \upharpoonright \mathcal{F}_n$, which exists for the class of potentials considered here, $n \in \mathbb{N}_0$. With the help of the particle number operator N, one then defines the renormalized Hamiltonian $H_r \doteq H + E(N)$. It is non-negative, the vacuum Ω is its ground state, and it induces the same time evolution on $\overline{\mathfrak{A}}$ as the given Hamiltonian H since N commutes with the elements of this algebra. A particularly nice class of potentials V are those of positive type (having non-negative Fourier transforms). They can have bound states as well. One easily verifies for these potentials that the corresponding renormalized Hamiltonians $H_r \doteq H + V(0)N$ are non-negative, where the value $V(0) > 0$ of the potential at the origin resembles a chemical potential. We will return to these potentials in our discussion of equilibrium states.

A more interesting class of states are approximate ground states, formed by a multitude of low energy Bosons, such as infra-vacua or condensates. We add here a few comments on the construction of such states in the present setting. For simplicity, we restrict attention to potentials V which are non-negative and of short range. We consider the situation, where the particle number is large, we have some information about localization properties of the states and their total energy is small.

In the first step of our construction we consider in the n-particle spaces $\mathcal{F}_n \subset \mathcal{F}$, $n \in \mathbb{N}$, product states of the form

$$\Psi_{L,n} \doteq (n!)^{-1/2} \, f_L \otimes_s \cdots \otimes_s f_L \, .$$

Here $f_L \in L^2(\mathbb{R}^s)$, $L > 0$, are normalized functions given by $\boldsymbol{x} \mapsto f_L(\boldsymbol{x}) = L^{-s/2} f(\boldsymbol{x}/L)$. We want to interpret these states as final configurations at large times of an approximate ground state. Making use of the Møller (wave) operators $\widehat{\Omega}_n$, which for the potentials considered here are limits of the Dyson cocycles $\Gamma_n(t)$ at asymptotic times $t \in \mathbb{R}$, cf. [13], this leads us to proceed to the vectors given by $\widehat{\Psi}_{L,n} \doteq \widehat{\Omega}_n \, \Psi_{L,n}$. What matters here is the fact that the isometric Møller operators intertwine the interacting and non-interacting dynamics, $e^{itH_n} \, \widehat{\Omega}_n = \widehat{\Omega}_n \, e^{itH_{0n}}$, $t \in \mathbb{R}$. Hence we obtain for the expectation value of the full energy operator the bounds

$$0 \leq \langle \widehat{\Psi}_{L,n}, \, H_n \, \widehat{\Psi}_{L,n} \rangle = \langle \Psi_{L,n}, \, H_{0,n} \, \Psi_{L,n} \rangle = nL^{-2} \int d\boldsymbol{x} \, |\boldsymbol{\partial} f(\boldsymbol{x})|^2 \, .$$

The functionals $\widehat{\omega}_{L,n}(\, \cdot \,) \doteq \langle \widehat{\Psi}_{L,n}, \, \cdot \, \widehat{\Psi}_{L,n} \rangle$ are approximate ground states of the interacting Hamiltonians in the limit $n \to \infty$, $nL^{-2} = \text{const}$. Note that the above integral can be made arbitrarily small for suitable choices of f. As a matter of fact, choosing functions whose Fourier transforms have compact support, all weak-*-limit points of these functionals, which exist by compactness arguments (Banach-Alaoglu theorem), lead to positive energy representations of the C*-dynamical system $(\overline{\mathfrak{A}}_\alpha, \boldsymbol{\alpha})$. More precisely, the automorphisms inducing the time translations are unitarily implemented in these representations and have positive generators, cf. [4, Sec. II.5].

Wheras the states $\widehat{\omega}_{L,n}(\, \cdot \,)$ approximate at asymptotic times Bose-Einstein condensates with particle density $\boldsymbol{x} \mapsto nL^{-s} |f(\boldsymbol{x}/L)|^2$, it is not quite clear whether they have such an interpretation also at finite times. In view of the interest in this phenomenon, cf. [12,17,22] and references quoted there, it seems worth while to explore this question also from the present point of view.

5.2 Asymptotic observables

In order to uncover from the algebra of field theoretic observables $\overline{\mathfrak{A}}$ specific particle observables, such as the momentum operators, one has to proceed to asymptotic times. For then the interaction between the fields fades away and the sub-leading particle aspects can surface. This fact was established first by Araki and Haag in relativistic quantum field theory [1,2], yet their reasoning can be carried over to the present non-relativistic setting.

In order to simplify the present discussion, we restrict our attention again to potentials V which are non-negative and of short range. Then bound states do not appear and since the theory is also asymptotically complete, cf. [13] and references quoted there, the isometric Møller operators are invertible.

Thus let $\widehat{\Omega}_n$ be the unitary Møller operator, mapping the states $\Psi_n \in \mathcal{F}_n$ onto outgoing scattering states, denoted by $\widehat{\Psi}_n \doteq \widehat{\Omega}_n \Psi_n$, $n \in \mathbb{N}_0$. One then has by the very definition of these operators the equality

$$\lim_{t \to \infty} \big(\langle \widehat{\Psi}_n , \, \boldsymbol{\alpha}(t)(A) \, \widehat{\Psi}_n \rangle - \langle \Psi_n , \, \boldsymbol{\alpha}^{(0)}(t)(A) \, \Psi_n \rangle \big) = 0 \, , \quad A \in \overline{\mathfrak{A}} \, .$$

Bearing in mind the structure of $\overline{\mathfrak{A}} \upharpoonright \mathcal{F}_n = \mathfrak{K}_n$ and the fact that the compact tensor factors appearing in these operators are mapped onto themselves by the non-interacting dynamics, one can show as in the proof of Lemma 3.4 that $\lim_{t \to \infty} \langle \Psi_n , \, \boldsymbol{\alpha}^{(0)}(t)(A) \, \Psi_n \rangle = \langle \Omega , \, A \, \Omega \rangle$. Since $\widehat{\Omega}_n$ maps each $\mathcal{F}_n$ onto itself, $n \in \mathbb{N}_0$, this implies that for any normalized vector $\Phi \in \mathcal{F}$ one has for the interacting dynamics

$$\lim_{t \to \infty} \langle \Phi , \, \boldsymbol{\alpha}(t)(A) \, \Phi \rangle = \langle \Omega , \, A \, \Omega \rangle \, , \quad A \in \overline{\mathfrak{A}} \, .$$

So, as expected, the asymptotically dominant contributions to the expectation values arise from the Fock vacuum.

For the next to leading contributions one proceeds to $A_0 \doteq A - \langle \Omega , \, A \, \Omega \rangle \, \mathbf{1}$, where one restricts attention to operators $A \in \overline{\mathfrak{A}}(\boldsymbol{O})$ which are localized in arbitrary bounded regions $\boldsymbol{O} \subset \mathbb{R}^s$. The operators $\boldsymbol{\alpha}(t, \boldsymbol{x})(A_0)$ register in the region $\boldsymbol{O} + \boldsymbol{x}$ at time t deviations from the Fock vacuum appearing in expectation values. Because of the spreading of wave packets, these deviations decay at asymptotic times. In order to compensate this effect one integrates the operators $\boldsymbol{\alpha}(t, \boldsymbol{x})(A_0)$ over regions in space whose diameter increases with time. In the proof that this can be done one makes use of the specific structure of the operators in $\overline{\mathfrak{A}}$ and arguments given in [8, Lem. 2.2], where the role of the energy operator is to be replaced by the particle number operator. In this way one can exhibit a dense set of localized operators $A \in \overline{\mathfrak{A}}$ such that for any test function h

$$\| \int \! d\boldsymbol{x} \, h(\boldsymbol{x}) \, \boldsymbol{\alpha}(\boldsymbol{x})(A_0) \upharpoonright \mathcal{F}_n \|_n \leq n \, c_A \, \| h \|_\infty \, , \quad n \in \mathbb{N}_0 \, ,$$

where $\| h \|_\infty$ denotes the supremum norm of h. Similarly to the leading order, it follows from this bound that

$$\lim_{t \to \infty} \big(\langle \widehat{\Psi}_n , \, \boldsymbol{\alpha}(t) (\int \! d\boldsymbol{x} \, h(\boldsymbol{x}/t) \, \boldsymbol{\alpha}(\boldsymbol{x})(A_0)) \, \widehat{\Psi}_n \rangle - \langle \Psi_n , \, \boldsymbol{\alpha}^{(0)}(t) (\int \! d\boldsymbol{x} \, h(\boldsymbol{x}/t) \, \boldsymbol{\alpha}(\boldsymbol{x})(A_0)) \, \Psi_n \rangle \big) = 0 \, .$$

By a routine computation of the second (non-interacting) term one gets

$$\lim_{t\to\infty} \langle \Psi_n,\, \boldsymbol{\alpha}^{(0)}(t)(\int d\boldsymbol{x}\, h(\boldsymbol{x}/t)\, \boldsymbol{\alpha}(\boldsymbol{x})(A_0))\, \Psi_n\rangle = c_s \int d\boldsymbol{p}\, h(2\boldsymbol{p})\langle \boldsymbol{p}|A_0|\boldsymbol{p}\rangle\, \langle \Psi_n, \widetilde{a}^*(\boldsymbol{p})\widetilde{a}(\boldsymbol{p})\Psi_n\rangle\,.$$

Here c_s is some dimension dependent constant, the tilde marks Fourier transformed quantities, and $\boldsymbol{p},\boldsymbol{q} \mapsto \langle \boldsymbol{p}|A_0|\boldsymbol{q}\rangle$ denotes the integral kernel of $A_0 \restriction \mathcal{F}_1$ in momentum space. This kernel is a continuous function for all compactly localized operators; its restriction to the diagonal $\boldsymbol{p} = \boldsymbol{q}$ is called sensitivity function of A_0, cf. [1,2]. The above relations can be combined into a single formula,

$$\lim_{t\to\infty} \int d\boldsymbol{x}\, h(\boldsymbol{x}/t)\, \boldsymbol{\alpha}(t,\boldsymbol{x})(A_0) = c_s \int d\boldsymbol{p}\, h(2\boldsymbol{p})\langle \boldsymbol{p}|A_0|\boldsymbol{p}\rangle\, \widehat{a}^*(\boldsymbol{p})\widehat{a}(\boldsymbol{p})\,,$$

where the limit exists on the domain of the number operator in $\mathcal{F}$ and $\widehat{a}^*$, $\widehat{a}$ denote the outgoing creation and annihilation operators in momentum space. An analogous formula holds at negative asymptotic times. Thus, from the field theoretic point of view, the particle momenta, desribed by the operators $\widehat{M} = \int d\boldsymbol{p}\, m(\boldsymbol{p})\, \widehat{a}^*(\boldsymbol{p})\widehat{a}(\boldsymbol{p})$ for the resulting family of functions m, become meaningful observables at asymptotic times. They can be determined from the underlying algebra in a universal manner which does not depend on the dynamics for the family of potentials considered here. Other important particle properties, such as the collision cross sections, can likewise be determined along these lines, cf. the discussion in [1,2,11].

The preceding results hold also for potentials admitting bound states. There the Fock space $\mathcal{F}$ splits at asymptotic times into the tensor product of Fock spaces corresponding to the elementary particle and to its stable bound states. The latter states can also be described in this manner in spite of their possibly complex internal structure. This internal structure is encoded in their sensitivity functions $\boldsymbol{p} \mapsto \langle \boldsymbol{p}, b|A_0|\boldsymbol{p}, b\rangle$, where b labels the bound states. Also in those cases, the leading asymptotic contributions to the expectation values of observables are described by the Fock vacuum. But in next to leading order, there appear in the above formula sums of terms containing the sensitivity functions and the creation and annihilation operators of all bound states in the theory.

These facts were explained here by relying on well known results about asymptotic completeness in quantum mechanics, cf. [13] and references quoted there. Yet it would be desirable to give more direct proofs. This may be possible by relying on the specific information about the properties of the kinematical observables, established in the present investigation, and a refinement of the Arveson spectral theory of automorphism groups [3]. What is needed there is a classification of the spectra in analogy to the measure classes for Hilbert space operators. For some progress in this direction, cf. [14].

5.3 Equilibrium states

The conventional method of constructing equilibrium states in quantum field theory is based on the consideration of Gibbs-von Neumann ensembles in bounded regions (boxes). In the present algebraic framework one deals from the outset with observables in infinite space, so one has to proceed differently. One replaces the sharp boundaries of a box by external confining forces which are conveniently described by a harmonic oscillator potential. Thus one considers Hamiltonians of the form, $L > 0$,

$$H_L \doteq \int d\boldsymbol{x} \big(\boldsymbol{\partial} a^*(\boldsymbol{x})\, \boldsymbol{\partial} a(\boldsymbol{x}) + (\boldsymbol{x}^2/L^4)\, a^*(\boldsymbol{x}) a(\boldsymbol{x})\big) + \int d\boldsymbol{x} \int d\boldsymbol{y}\; a^*(\boldsymbol{x}) a^*(\boldsymbol{y}) V(\boldsymbol{x}-\boldsymbol{y}) a(\boldsymbol{x}) a(\boldsymbol{y})\,,$$

where the interaction potential V is of positive type. Proceeding to the renormalized Hamiltonians $H_{L\,r} \doteq H_L + V(0)N$, the corresponding canonical ensembles can then be described by density matrices on Fock space $\mathcal{F}$. In fact, one has $H_{L\,r} \geq H_{0L}$, where H_{0L} is the Hamiltonian of the harmonic oscillator, arising for interaction potential $V = 0$. So the partition functions exist for $H_{L\,r}$ by the Golden-Thompson inequality.

Similarly to the Hamiltonian in equation (1.1), the unitary operators e^{itH_L} induce automorphisms $\boldsymbol{\alpha}_L(t) \doteq \mathrm{Ad}\, e^{itH_L}$, $t \in \mathbb{R}$, of the algebra $\overline{\mathfrak{A}}$, and Theorem 4.5 applies accordingly for any given L. Moreover, in the limit of large L, these automorphisms converge pointwise to the original dynamics $\boldsymbol{\alpha}(t)$, $t \in \mathbb{R}$, relative to the locally convex topology on $\overline{\mathfrak{A}}$. Proofs of these statement are given in the appendix.

For the construction of equilibrium states in the thermodynamic limit, one picks any $L > 0$ and considers for given $\beta > 0$ and $\mu \geq V(0)$ the family of states on $\overline{\mathfrak{A}}$

$$\omega_{\beta,\mu,L}(\,\cdot\,) \doteq \mathrm{Tr}\,(e^{-\beta(H_L+\mu N)}\,\cdot\,)/\mathrm{Tr}\, e^{-\beta(H_L+\mu N)}\,.$$

These states satisfy the KMS-condition at inverse temperature β and chemical potential μ for the dynamics $\boldsymbol{\alpha}_L(t)$, $t \in \mathbb{R}$, so they describe equilibria [5]. Moreover, as L approaches infinity, they have weak-*-limit points $\{\omega_{\beta,\mu}\}$ on $\overline{\mathfrak{A}}$ by the Banach-Alaoglu Theorem. The limit states need neither be unique, as is typically the case in the presence of phase transitions. Nor need they describe pure phases, such as in the presence of spontaneous breakdown of symmetries, where mixtures of phases can appear.

It is also not clear from the outset that the limit states describe equilibria for the original dynamics $\boldsymbol{\alpha}(t)$, $t \in \mathbb{R}$. In case of no interaction, one can show that the limit states obtained this way agree with the familiar quasi-free KMS states obtained in the thermodynamic limit, cf. [5, Sec. 5.2.5]; a description of quasi-free states in the framework of the resolvent algebra can be found in [9, Sec. 4]. In the presence of interaction some more detailed analysis of the limit states is required, however. There one can rely on methods developed for C*-dynamical system, such as $(\overline{\mathfrak{A}}_\alpha, \alpha)$, cf. [5] and references quoted there.

The present formalism provides also a basis for the discussion of the effects of perturbations of KMS states and their return to equilibrium. Moreover, since one is dealing with the theory in infinite space, it ought to cover states where the translation symmetry is spontanteously broken, such as crystals, or states in motion, such as fluids. There exists an extensive literature on these topics and we refrain from giving references here; comprehensive lists may be found in [5, 20].

We conclude this section by noting that one can also study in the present framework the formation of Bose-Einstein condensates, trapped by a harmonic potential, cf. [12, 17, 22]. There arises the interesting question whether the condensation phenomenon disappears if the external potential is turned off, *i.e.* whether the limit states agree on $(\overline{\mathfrak{A}}_\alpha, \alpha)$ with the Fock vacuum. This is likely to be the case in the Gross-Pitaevskii model because of the assumed repulsive interaction. Yet for interaction potentials of positive type, admitting bound states, some more interesting limit states might appear.

6 Summary and outlook

In the present article we have established a consistent algebraic framework for the treatment of interacting non-relativistic Bose fields. The novel feature of our approach consists of the fact that "large field problems" are avoided from the outset by dealing with the resolvents of the fields. Thus in singular states, describing accumulations of particles with infinite density, these operators simply vanish. On the mathematical side, this implies that the resolvent algebra has ideals; but this is inevitable if a kinematical algebra is to admit a sufficiently rich family of different dynamics [9, Sec. 10.18]. As a consequence of our approach, the algebra of gauge invariant observables attains a mathematically convenient structure, being the inverse limit of a family of approximately finite dimensional C*-algebras. The dynamics considered in the present investigation, describing attractive and repulsive interactions, act by automorphisms on this algebra in a continuous and quasilocal manner. Hence interacting Bosons can be described by C*-dynamical systems, contrary to apparent obstructions conceived in the literature [5, 18].

The present results also shed light on the relation between the field theoretic setting and the quantum mechanical particle picture, which is based on position and momentum operators. Whereas the field-theoretic observable algebra is built from operators of finite rank, the particle algebra generated by position and momentum operators accomodates an abundance of operators with continuous spectrum. Thus from the present point of view it seems advantageous to first construct the states of interest in the field theoretic setting and only then turn to their analysis and physical interpretation.

In order to illustrate this idea, we have briefly discussed some standard problems in many body theory. We have sketched how the existence of ground states, including approximate ground states (infra-vacua), can be established in the present framework. Particle features are uncovered by proceeding to asymptotic times. Suitable spatial averages of localized observables converge in this limit to operators which describe the asymptotic particle momenta. In a similar manner one can also compute collision cross sections, *etc.* In order to explain these facts we made use of well known results on asymptotic completeness in quantum mechanics. Yet it seems that a more direct approach is possible in the present algebraic setting by a refinement of the Arveson spectral theory of automorphism groups [3].

Finally, we have addressed the problem of constructing thermal equilibrium states for pair potentials of positive type, allowing for bound states of Bosons. There one is profitting from the fact that the dynamics of a thermal system, which is trapped by an external harmonic potential, also acts by automorphisms on the algebra of observables. Turning off the external potential, these automorphisms converge on all observables to the dynamics of the infinite system. Moreover, thermodynamic limits of the trapped equilibrium states exist. Yet these limit states may neither be unique nor need they necessarily describe equilibria. So some more detailed analysis is required in order to clarify their specific properties in each particular case. In view of the present results, one can rely there on the powerful methods developed for C*-dynamical systems.

Another topic of interest consists of the extension of the present analysis to the full resolvent algebra of Bose fields. It seems appropriate to proceed there also to a slightly larger algebra, containing the operators $X_f \doteq a^*(f)\,(1 + a^*(f)a(f))^{-1/2}$, $f \in \mathcal{D}(\mathbb{R}^s)$, and their adjoints. These operators are isometries if f is normalized. They are limits of the operators $X_{f,\kappa} \doteq a^*(f)\,(1+a^*(f)a(f))^{-\kappa}$, $\kappa > 1/2$, contained in the resolvent algebra of fields. Gauge invariant combinations of these operators of the form $X_f X_g^*$, where $f, g \in \mathcal{D}(\mathbb{R}^s)$, appear already as elements of the algebra of observables. We conjecture that the partially time translated operators $\boldsymbol{\alpha}(t)(X_f)\,X_g^*$ and $X_f\,\boldsymbol{\alpha}(t)(X_g^*)$, $t \in \mathbb{R}$, are also elements of this algebra, at least for pair potentials of positive type. A proof of this conjecture would provide evidence to the effect that the dynamics considered here act by automorphisms on the full resolvent algebra of fields.

Let us mention in conclusion that the present ideas lead also to simplifications in the analysis of fermionic systems. There one can show by similar arguments that the dynamics considered here act by automorphisms on the associated subalgebras of observables as well. Similar results can be deduced from the work of Narnhofer and Thirring on the dynamics of Fermi fields [18]. Yet, for technical reasons, these authors considered pair potentials with an ultraviolet cutoff. This restriction can be removed, however, one can proceed there as well to the generic class of potentials considered in the present investigation.

Appendix

We establish here results stated in the main text in the proof of Lemma 4.1 and in Subsection 5.3. Given $n \in \mathbb{N}$, we consider on the unsymmetrized Hilbert space $\mathcal{H}_n$ the Hamiltonians

$$H_{nL} \doteq \sum_i (\boldsymbol{P}_i^2 + \boldsymbol{Q}_i^2/L^4) + \sum_{j \neq k} V(\boldsymbol{Q}_j - \boldsymbol{Q}_k)$$

for $L > 0$, respectively $L = \infty$ (no external forces). These Hamiltonians augment the Hamiltonian H_n, given in equation (2.1), by an external harmonic potential, where we identify $H_{n\,\infty} = H_n$. It follows from standard results, cf. [21, Sec. VIII.7], that $H_{nL} \to H_n$ in the strong resolvent sense as $L \to \infty$ and consequently $e^{itH_{nL}} \to e^{itH_n}$ in the strong operator topology, uniformly on compact subsets of $t \in \mathbb{R}$.

The adjoint action of the unitaries $e^{itH_{nL}}$ on $\mathcal{B}(\mathcal{H}_n)$ is denoted by $\alpha_{nL}(t) \doteq \mathrm{Ad}\, e^{itH_{nL}}$, where we identify $\alpha_{n\,\infty}(t) = \alpha_n(t)$, $t \in \mathbb{R}$. Each group of automorphisms $\alpha_{nL}(t)$, $t \in \mathbb{R}$, leaves the algebra $\mathfrak{K}(\mathbb{I}_n) \subset \mathcal{B}(\mathcal{H}_n)$ invariant and acts pointwise norm-continuously on it. This is apparent for the dynamics $\alpha_{nL}^{(0)}(t)$, $t \in \mathbb{R}$, where the interaction potential is put equal to $V = 0$. In order to see that this assertion is also true in the interacting case, we proceed as in the main text and consider the Dyson cocycles $\gamma_{nL}(t) \doteq \alpha_{nL}(t) \circ \alpha_{nL}^{(0)}(-t)$, $t \in \mathbb{R}$, and their expansion in terms of multiple integrals, similarly to equation (4.2). The basic ingredient in this expansion are the derivations, defined as in Lemma 4.1 by

$$\boldsymbol{\delta}_{nL}(s)(B) \doteq \int_0^s du\, [B, \boldsymbol{V}_{nL}(u)]\,, \quad B \in \mathcal{B}(\mathcal{H}_n)\,. \tag{6.1}$$

Here $\boldsymbol{V}_{nL}(u) \doteq \boldsymbol{\alpha}_{nL}^{(0)}(u)(\boldsymbol{V}_n)$, where $\boldsymbol{V}_n = \sum_{j \neq k} V(\boldsymbol{Q}_j - \boldsymbol{Q}_k)$. Making use of the notation in the proof of Lemma 4.1, we put $\boldsymbol{Q}_{j \frown k} \doteq \boldsymbol{Q}_j - \boldsymbol{Q}_k$, $\boldsymbol{P}_{j \frown k} \doteq \boldsymbol{P}_j - \boldsymbol{P}_k$, and note that for the individual terms appearing in the time shifted sum we obtain

$$V_{jk\,L}(u) \doteq \boldsymbol{\alpha}_{nL}^{(0)}(u)(V(\boldsymbol{Q}_{j\frown k})) = V(c_L(u)\,\boldsymbol{Q}_{j\frown k} + s_L(u)\,\boldsymbol{P}_{j\frown k}) \in \mathfrak{R}_n(j \frown k)\,.$$

Here $u \mapsto c_L(u) \doteq \cos(2u/L^2)$, $u \mapsto s_L(u) \doteq L^2 \sin(2u/L^2)$ for finite L and $u \mapsto c_\infty(u) \doteq 1$, $u \mapsto s_\infty(u) \doteq 2u$ for $L = \infty$. We make use again of the fact that the derivations $\boldsymbol{\delta}_{nL}(s)$ in equation (6.1) are finite sums of derivations $\delta_{jkL}(s)$, where the potential $\boldsymbol{V}_{nL}$ in equation (6.1) is replaced by the pair potential $V_{jk\,L}$.

As was explained in Lemma 4.1, an important step in the analysis of the derivations $\delta_{jkL}(s)$ consists of the demonstration that the underlying integrals $\int_0^s du V_{jk\,L}(u)$, defined in the strong operator topology, belong to the compact ideal $\mathfrak{C}_n(j \frown k) \subset \mathfrak{R}_n(j \frown k)$. Since $\mathfrak{R}_n(j \frown k)$ is faithfully represented on $L^2(\mathbb{R}^s)$, it is sufficient to show that the above integrals act there as compact operators. Now the potential V is an element of $C_0(\mathbb{R}^s)$, so the product $V_{jk\,L}(u)V_{jk\,L}(v)$ is, for any given L, a compact operator on $L^2(\mathbb{R}^s)$ whenever $u, v \in \mathbb{R}$ satisfy $h_L(u,v) \doteq (s_L(u)c_L(v) - s_L(v)c_L(u)) \neq 0$. For then the two underlying time shifted position operators are canonically conjugate with Planck constant $h_L(u,v)$; see below for a more explicit argument.

It follows, disregarding a set of measure zero, that the function $u, v \mapsto V_{jk\,L}(u)V_{jk\,L}(v)$ on $\mathbb{R} \times \mathbb{R}$ has values in the compact operators. Moreover, it is bounded in norm. This implies that the double integral $\int_0^s du \int_0^s dv\, V_{jk\,L}(u)V_{jk\,L}(v) = (\int_0^s du\, V_{jk\,L}(u))^2$ is a compact operator for any $s \in \mathbb{R}$. Hence its square root is also compact, so by polar decomposition we arrive at the conclusion that $\int_0^s du\, V_{jk\,L}(u)$ is a compact operator for any $s \in \mathbb{R}$ and any choice of L.

As in the proof of Lemma 4.1, one then shows that the algebra $\mathfrak{K}(\mathbb{I}_n)$ is stable under the action of the derivations $\boldsymbol{\delta}_{nL}(s)$. The function $s \mapsto \boldsymbol{\delta}_{nL}(s)$ acts norm-continuously on this algebra and is bounded on compact subsets of $s \in \mathbb{R}$. Thus one arrives by arguments given in Sec. 4 at a generalization of Proposition 4.3 for the Hamiltonians H_{nL}, where $L > 0$ or $L = \infty$. This completes the proof of the above statement on the automorphic action of dynamics on the algebra $\mathfrak{K}(\mathbb{I}_n)$ with or without external harmonic forces.

Next, we adress the question regarding the convergence properties of the dynamics $\alpha_{nL}(t)$ in the limit of large L. In the non-interacting case, $V = 0$, the corresponding dynamics $\alpha_{nL}^{(0)}(t)$, $L > 0$, leave each subalgebra $\mathfrak{C}_n(\mathbb{I}_m) \subset \mathfrak{K}(\mathbb{I}_n)$ invariant, cf. the discussion after Lemma 4.1. It therefore follows from the strong operator convergence of the underlying unitary operators, mentioned above, that $\alpha_{nL}^{(0)}(t) \to \alpha_n^{(0)}(t)$ pointwise in norm on $\mathfrak{K}(\mathbb{I}_n)$ as L tends to infinity, $t \in \mathbb{R}$.

In order to establish this fact also for the interacting dynamics, we need to compare the expansions of the Dyson cocyles for different choices of L. This is accomplished by analyzing the norm distance between the operators $\int_0^s du V_{jk\,L}(u)$, $s \in \mathbb{R}$, for different values of L. It requires a refinement of the preceding arguments. As above, we consider the function

$$u, v \mapsto V_{jk\,L}(u) V_{jk\,L}(v) = e^{iuH_{1L}} V(\boldsymbol{Q}_{j\frown k})\, e^{i(v-u)H_{1L}}\, V(\boldsymbol{Q}_{j\frown k})\, e^{-ivH_L}\,,$$

where $H_{1L} \doteq \boldsymbol{P}^2_{j\frown k} + \boldsymbol{Q}^2_{j\frown k}/L^4$. Adopting the Dirac bra-ket notation, the kernel of the middle term of this operator is given in configuration space by

$$\begin{aligned}
&\boldsymbol{x},\boldsymbol{y} \to \langle \boldsymbol{x}|\, V(\boldsymbol{Q})\, e^{i(v-u)H_{1L}}\, V(\boldsymbol{Q})\, |\boldsymbol{y}\rangle \\
&\quad = i\big(2\pi i L^2 \sin(2(u-v)/L^2)\big)^{-s/2}\, V(\boldsymbol{x})\, e^{i((\boldsymbol{x}^2+\boldsymbol{y}^2)\cos(2(u-v)/L^2) - 2\boldsymbol{x}\boldsymbol{y})/2L^2 \sin(2(u-v)/L^2)}\, V(\boldsymbol{y})\,,
\end{aligned}$$

where we used the Green's function (Mehler kernel) of the Hamiltonian H_{1L}. Choosing for the potential V some test function, this kernel is square integrable for $(u-v) \neq \pi L^2\,\mathbb{Z}$. Thus, for these u, v, the operator function $u, v \mapsto V(\boldsymbol{Q})\, e^{i(v-u)H_{1L}}\, V(\boldsymbol{Q})$ has values in the Hilbert-Schmidt class. Moreover, it is continuous with regard to the Hilbert-Schmidt norm and converges to $u, v \mapsto V(\boldsymbol{Q})\, e^{i(v-u)H_1}\, V(\boldsymbol{Q})$ for $L \to \infty$. Since any $V \in C_0(\mathbb{R}^s)$ can be approximated in norm by testfuctions, the preceding statements remain true for any such potential if one replaces the terms "Hilbert-Schmidt class" by "compact operators" and "Hilbert-Schmidt norm" by "operator norm", denoted by $\|\cdot\|_1$. Thus the function

$$u, v \mapsto \|\, V(\boldsymbol{Q})\, e^{i(v-u)H_{1L}}\, V(\boldsymbol{Q}) - V(\boldsymbol{Q})\, e^{i(v-u)H_1}\, V(\boldsymbol{Q})\|_1$$

is bounded for $u, v \in \mathbb{R}$, continuous for $(u-v) \neq \pi L^2\,\mathbb{Z}$, and it tends to 0 for these u, v in the limit $L \to \infty$. Next, since for $u \neq v$ the operators $V(\boldsymbol{Q})\, e^{i(v-u)H_1}\, V(\boldsymbol{Q})$ are compact and $e^{iuH_{1L}} \to e^{iuH_1}$ in the strong operator topology for large L, the function

$$u, v \mapsto \|(e^{iuH_{1L}} - e^{iuH_1}) V(\boldsymbol{Q})\, e^{i(v-u)H_1}\, V(\boldsymbol{Q})\|_1$$

has the same properties as the preceding one. This applies also to the corresponding function of the adjoint operators. It is straightforward to see that these facts imply

$$\begin{aligned}
&\lim_{L\to\infty} \| \int_0^s du \int_0^s dv\, (V_{jk\,L}(u) V_{jk\,L}(v) - V_{jk}(u) V_{jk}(v))\|_1 \\
&\leq \int_0^s du \int_0^s dv\, \lim_{L\to\infty} \|(V_{jk\,L}(u) V_{jk\,L}(v) - V_{jk}(u) V_{jk}(v))\|_1 = 0\,,
\end{aligned}$$

making use of the dominated convergence theorem. In a similar vein one obtains

$$\lim_{L\to\infty} \| \int_0^s du\, (V_{jkL}(u) - V_{jk}(u)) \int_0^s dv\, V_{jk}(v) \|_1 = 0\,,$$
$$\lim_{L\to\infty} \| \int_0^s du\, V_{jk}(u) \int_0^s dv\, (V_{jkL}(v) - V_{jk}(v)) \|_1 = 0\,,$$

since one of the integrals appearing in these relations is a compact operator and the other one converges to 0 in the strong operator topology. Putting everything together, we obtain

$$\begin{aligned}
&\lim_{L\to\infty} \| \int_0^s du\, V_{jk\,L}(u) - \int_0^s du\, V_{jk}(u) \|_1^2 \\
&\quad = \lim_{L\to\infty} \| \int_0^s du \int_0^s dv\, (V_{jk\,L}(u) - V_{jk}(u))(V_{jk\,L}(v) - V_{jk}(v)) \|_1 = 0\,.
\end{aligned}$$

Thus the derivations $s \to \boldsymbol{\delta}_{nL}(s) = \sum_{j\neq k} \delta_{jk\,L}(s)$, generating the Dyson cocycles, are norm continuous in $s \in \mathbb{R}$, uniformly in L, and they satisfy $\lim_{L\to\infty\, n}\|\boldsymbol{\delta}_{nL}(s) - \boldsymbol{\delta}_n(s)\| = 0$. Here ${}_n\| \cdot \|$ denotes the norm of the linear maps on $\mathcal{B}(\mathcal{H}_n)$. Now the norm difference ${}_n\|\gamma_{nL}(t) - \gamma_n(t)\|$ of the Dyson cocycles, defined in relation (4.2), is bounded from above by a (uniformly with regard to L) convergent sum over m, l of terms of the form, $1 \leq m \leq l$,

$${}_n\| \int_0^t ds_l \ldots \int_0^{s_2} ds_1\, \boldsymbol{\delta}_{nL}(s_l) \circ \cdots \circ \boldsymbol{\delta}_{nL}(s_{m+1}) \circ \big(\boldsymbol{\delta}_{nL}(s_m) - \boldsymbol{\delta}_n(s_m)\big) \circ \boldsymbol{\delta}_n(s_{m-1}) \cdots \circ \boldsymbol{\delta}_n(s_1) \|\,.$$

Hence, by the dominated convergence theorem, $\lim_{L\to\infty\, n}\|\gamma_{nL}(t) - \gamma_n(t)\| = 0$. So, bearing in mind the remarks about the non-interacting dynamics, one finds that $\alpha_{nL}(t) \to \alpha_n(t)$ pointwise in norm on each algebra $\mathfrak{K}(\mathbb{I}_n)$, $n \in \mathbb{N}$, as L tends to infinity, $t \in \mathbb{R}$.

Arguing now as in the second part of Sec. 4, one concludes that for all $L > 0$ the corresponding automorphism groups $\boldsymbol{\alpha}_L(t)$, $t \in \mathbb{R}$, leave the algebra of observables $\overline{\boldsymbol{\mathfrak{A}}}$ invariant and act pointwise continuouly on it, so all statements of Theorem 4.5 apply to them. Moreover, $\boldsymbol{\alpha}_L(t) \to \boldsymbol{\alpha}(t)$, $t \in \mathbb{R}$, pointwise in the limit of large L with regard to the locally convex topology on the observable algebra $\overline{\boldsymbol{\mathfrak{A}}}$. This completes the proof of the assertions in the main text.

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