Split property for free
massless finite helicity fields

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Abstract

We prove the split property for any finite helicity free quantum fields. Finite helicity Poincaré representations extend to the conformal group $\mathcal{C}$ (cf. [18]) and the conformal covariance plays an essential role in the argument: the split property is ensured by the trace class condition $\mathrm{Tr}\ (e^{-\beta L_0}) < +\infty$ for the conformal Hamiltonian L_0 of the Möbius covariant restriction of the net on the time axis. We extend the argument for the scalar case presented in [7]. We provide the direct sum decomposition into irreducible representations of the conformal extension of any helicity-h representation to the subgroup of transformations fixing the time axis. Our analysis provides new relations among finite helicity representations and suggests a new construction for representations and free quantum fields with non-zero helicity.

1 Introduction

The split property in quantum field theory can be viewed as a strong version of locality. Locality (= Einstein causality) requires the bounded observables localized in two spacelike separated regions O_1 and O_2 to generate two commuting von Neumann algebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$. The split property demands that the algebra generated by $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ is naturally isomorphic to the tensor product $\mathcal{A}(O_1) \otimes \mathcal{A}(O_2)$ and this can hold only if there

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is some finite positive distance between the regions O_1 and O_2, due to UV singularities that arise when the regions touch.

Physically, the split property is motivated as a "statistical indepence" in the sense that states can be independently prepared in O_1 and O_2: for every pair of normal states on $\mathcal{A}(O_i)$, there is a normal state of the full QFT that on $\mathcal{A}(O_i)$ coincides with the given states [5]. The relative tensor product position is also an indispensible prerequisite without which a notion of entanglement of states between the two subsystems cannot be defined [15].

The terminology "split" for a pair of commuting algebras actually refers to the inclusion of one algebra in the commutant of the other, asserting that there exists a type I factor[1] $\mathcal{B}$ such that

$$\mathcal{A}_1 \subset \mathcal{B} \subset \mathcal{A}'_2,$$

cf. Definition 5.1. Because local algebras in QFT are in general type III (a characteristic feature of QFT as compared to quantum mechanical systems), the split property is not ensured by basic assumptions.

Whether the split property holds for two local algebras at a finite distance, is a feature of the QFT model under consideration. It has been verified in various models in quantum field theory, see, e.g., [8, 20, 7]. Several sufficient conditions are known in terms of the trace-class property of certain operators related to phase space [8, 6, 7], indicating that typically, "too many degrees of freedom" may cause it to fail. A deep mathematical understanding of the split property was given in [11].

For the massless scalar free field in four spacetime dimensions, the split and nuclearity properties for an inclusion of non-touching double cone regions has been established in [7]. The argument is essentially group theoretic: the one-particle space of the massless free field carries an irreducible representation U of the Poincaré group that extends to the conformal group $\mathcal{C}$ in four dimensions. $\mathcal{C}$ is the 15-dimensional Lie group generated by the Poincaré group and the "conformal inversion" I, cf. (2.1). It contains the dilations and the special conformal transformations $I \circ t \circ I$, where t is a translation.

The 3-dimensional subgroup generated by time translations and the conformal inversion is isomorphic to the Möbius group $\mathsf{Möb} = \mathrm{SL}(2, \mathbb{R})/\mathbb{Z}_2$, and acts geometrically on the time axis $\vec{x} = 0$ exactly like the conformal symmetry group of a chiral conformal QFT. This means that a conformal quantum field theory in four dimensions, when restricted to the time axis, becomes a chiral conformal QFT. In the scalar case, the chiral currents of this theory are the free scalar field restricted to the time axis, along with all its spatial derivatives $\nabla_{a_1} \ldots \nabla_{a_k} \varphi(t, 0)$. Their scaling dimensions increase with the number of spatial derivatives.

The number of quasiprimary (i.e., $\mathsf{Möb}$-covariant) currents as a function of their scaling dimension is controlled by representation theory. In this way, the authors of [7] could establish that the operator $e^{-\beta L_0}$ has a finite trace, where L_0 is the conformal Hamiltonian of this chiral conformal QFT.

This suffices to establish the split property for the algebra inclusions $\mathcal{A}(O) \subset \mathcal{A}(\tilde{O})$ when $O \Subset \tilde{O}$ are two double cones with apices on the time axis. This implies the statistical independence of $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ whenever $O_1 = O$ and O_2 is contained in the causal complement of $\tilde{O}$, and then, by covariance, whenever O_1 and O_2 are spacelike separated double cones with a finite distance.

We adapt this argument to all massless free field theories of finite helicity, including the free Maxwell field. Because $\mathsf{Möb}$ commutes with the subgroup $\mathrm{SO}(3)$ of spatial rotations,

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[1]i.e., isomorphic to $\mathcal{B}(\mathcal{H})$ of some Hilbert space $\mathcal{H}$.

the proof reduces to the computation of the restriction of the unitary representation of the conformal group on the one-particle space to the subgroup $\mathsf{M\ddot{o}b} \times \mathrm{SO}(3)$, where the representations of $\mathrm{SO}(3)$ just provide multiplicities for the irreducible representations of $\mathsf{M\ddot{o}b}$. The traces of $e^{-\beta L_0}$ in irreducible representations of $\mathsf{M\ddot{o}b}$ are well known, and the trace-class property on the one-particle space is obtained by an explicit computation. This also implies the L^2-nuclearity property.

Our computation leads to a remarkable observation: as a representation of $\mathsf{M\ddot{o}b} \times \mathrm{SO}(3)$, the one-particle space for helicity $h + 1$ is just a subrepresentation of that for helicity h (provided $h > 0$). This suggests some new kind of "deformation argument" to construct helicity $h + 1$ from helicity h, cf. Sect. 6.

2 Preliminaries

2.1 Minkowski spacetime and the Poincaré group

Let $\mathbb{R}^{1+3}$ be Minkowski space, i.e., $\mathbb{R}^4$ endowed with the metric

$$(x, y) = x_0 y_0 - \sum_{i=1}^{3} x_i y_i.$$

In a 4-vector $x = (x_0, x_1, x_2, x_3)$, $x_0 = t$ and $\vec{x} = \{x_i\}_{i=1,2,3}$ are the time and space coordinates, respectively. The metric induces a causal structure, in particular the future $x + V_+$ of a point x, where $V_+ = \{y \in \mathbb{R}^4 : (y, y) > 0, y_0 > 0\}$. The causal complement of a region O is given by $O' = \{x \in \mathbb{R}^{1+3} : (x - y, x - y) < 0, \forall y \in O\}$. A **causally closed** region is such that $O = O''$. Particularly nice causally closed regions are the open double cones of the form $O = x_- + V_+ \cap x_+ - V_+$, where x_+ is a point in the future of x_-.

The **Poincaré group** $\mathcal{P}$ is the inhomogeneous symmetry group of $\mathbb{R}^{1+3}$. It is the semidirect product of the Lorentz group $\mathcal{L}$, the homogeneous Minkowski symmetry group, and the translation group $\mathbb{R}^4$, i.e., $\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L}$. We shall indicate with $\mathcal{P}_+^{\uparrow} = \mathbb{R}^4 \rtimes \mathcal{L}_+^{\uparrow}$ the connected component of the identity, with $\widetilde{\mathcal{P}}_+^{\uparrow}$ and $\widetilde{\mathcal{L}}_+^{\uparrow}$ the universal coverings resp. of $\mathcal{P}_+^{\uparrow}$ and $\mathcal{L}_+^{\uparrow}$, and with Λ the covering map.

The conformal group $\mathcal{C}$ in four spacetime dimensions is the extension of the Poincaré group by the "conformal inversion"

$$I : (t, \vec{x}) \mapsto \frac{(-t, \vec{x})}{t^2 - \vec{x}^2}. \tag{2.1}$$

Notice that I is singular on $\mathbb{R}^{1+3}$, but one can extend Minkowski space to the "Dirac manifold" on which $\mathcal{C}$ acts without singularities, and of which Minkowski space is a dense chart. $\mathcal{C}$ is a 15-dimensional Lie group isomorphic to $\mathrm{SO}(2, 4)_0$. The minus sign in the t-component of (2.1) ensures that I belongs to the connected component.

2.2 Massless representations of the Poincaré group

The characters of the translation group are $a \mapsto e^{i(a,q)}$ where $q \in \mathbb{R}^4$ is a momentum. According to Wigner [31], irreducible positive-energy representations of $\widetilde{\mathcal{P}}_+^{\uparrow}$ are induced by what is now called Mackey induction, from irreducible representations of the stabilizer subgroup (also known as the "little group") of some q appearing in the representation. The

characters appearing in massless positive-energy representations of $\widetilde{\mathcal{P}}_+^\uparrow$ are given by $q \neq 0$ contained in $\partial V_+ = \{x \in \mathbb{R}^{1+3} : (x,x) = 0, x_0 \geq 0\}$. We fix

$$q \equiv (1,0,0,1) \in \partial V_+$$

($\partial V_+ \setminus \{0\}$ is a $\mathcal{L}_+^\uparrow$-orbit). We shall call Stab and $\overline{\mathrm{Stab}}_q$ the stabilizers of the point q through the $\widetilde{\mathcal{L}}_+^\uparrow$ and $\widetilde{\mathcal{P}}_+^\uparrow$ actions, respectively. The latter is the semidirect product of $\mathbb{R}^{3+1}$ and the little group Stab_q, i.e. $\overline{\mathrm{Stab}}_q = \mathbb{R}^4 \rtimes \mathrm{Stab}_q$. Any massless $\widetilde{\mathcal{P}}_+^\uparrow$ unitary positive energy representation is obtained inducing by a unitary representation of the $\overline{\mathrm{Stab}}_q$ group. Note that a $\overline{\mathrm{Stab}}_q$ representation is of the form $\mathbb{R}^4 \rtimes \mathrm{Stab}_q \ni (x,\sigma) \mapsto V(\sigma)q(x)$ where V is the unitary representation of Stab_q.

The little group Stab_q is isomorphic to $\widetilde{E}(2)$, the double cover of the Euclidean group of the two dimensional euclidean space, namely $E(2) = \mathbb{R}^2 \rtimes \mathbb{T}$. Let $U = \mathrm{Ind}_{\overline{\mathrm{Stab}}_q \uparrow \widetilde{\mathcal{P}}_+^\uparrow} V q$ be a unitary representation of $\widetilde{\mathcal{P}}_+^\uparrow$ induced from the representation Vq of $\overline{\mathrm{Stab}}_q$. We say, in case V trivial on $\tilde{E}(2)$-translations, that U has *finite helicity* (or finite spin); in the other cases has *infinite spin*.

An irreducible finite helicity representation is of the form

$$U_h = \mathrm{Ind}_{\overline{\mathrm{Stab}}_q \uparrow \widetilde{\mathcal{P}}_+^\uparrow} V_h\, q, \qquad h \in \frac{1}{2}\mathbb{Z}$$

where $V_h(g,x) = h(g)$ where h is the one dimensional representation of the double covering of $\mathbb{T}$ of character $2h \in \mathbb{Z}$ (V_h has to be trivial on $\widetilde{E}(2)$ translation subgroup). h is called *helicity* parameter.

Massless representations of $\widetilde{\mathcal{P}}_+^\uparrow$ of finite helicity extend to unitary representations $\widetilde{U}$ of the conformal group $\mathcal{C}$. The main argument in our paper pertains to the restriction of this extension to the Möbius subgroup of $\mathcal{C}$ (Sect. 2.3). We denote by P_μ the generators of the translations, and $K_\mu := \widetilde{U}(I)P^\mu\widetilde{U}(I)$ the generator of the special conformal transformations. Then $i[P_\mu, K_\nu] = -2\eta_{\mu\nu}D + 2M_{\mu\nu}$ where D and $M_{\mu\nu}$ are the generators of the dilations and Lorentz transformations. The conformal Hamiltonian $L_0 = \frac{1}{2}(P_0 + K_0)$ generates the rotations in $\mathrm{SO}(2) \oplus \mathbf{1}_4 \subset \mathrm{SO}(2,4)_0 \simeq \mathcal{C}$, and $\widetilde{U}(I) = e^{i\pi L_0}$.

2.3 The Möbius group and its representations

The Möbius group. The Möbius group Möb is the 3-dimensional Lie group $\mathrm{PSU}(1,1) = \mathrm{SU}(1,1)/\mathbb{Z}_2$ acting on $S^1 \subset \mathbb{C}$ by fractional linear transformations

$$S^1 \ni z \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}, \qquad \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1).$$

Via the Cayley transform and its inverse, the stereographic projection:

$$C : \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \ni x \mapsto -\frac{x-i}{x+i} \in S^1, \qquad C^{-1} : S^1 \ni z \mapsto -i\frac{z-1}{z+1} \in \overline{\mathbb{R}}$$

it is isomorphic to $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$ acting on the compactified real line $\overline{\mathbb{R}}$ via

$$\overline{\mathbb{R}} \ni x \mapsto \frac{ax+b}{cx+d} \in \overline{\mathbb{R}}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

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We shall freely switch between the "circle picture" and the "line picture".

Möb arises as the subgroup of the conformal group $\mathcal{C}$ in four spacetime dimension, generated by time translations and the conformal inversion (2.1). It preserves the time axis and commutes with SO(3), the spatial rotations. Its more familiar appearance in quantum field theory is in the (unbroken) conformal group in two spacetime dimension, that is isomorphic to Möb $\times$ Möb acting on the two-dimensional Dirac manifold $S^1 \times S^1$ where each S^1 is the compactification of one lightlike axis.

Möb can be generated by various one-parameter subgroups. Firstly, consider the following subgroups:

- Rotations $r : [0, 2\pi] \ni \theta \mapsto e^{i\theta}z \in S_1$, in the circle picture;

- Dilations $\delta : \mathbb{R} \ni s \mapsto e^s x \in \mathbb{R}$, in the line picture.

- Translations $t : \mathbb{R} \ni s \mapsto x + s \in \mathbb{R}$, in the line picture.

They are respectively denoted with $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{N}$. Any element $g \in$ Möb can be uniquely decomposed following the $\mathbf{KAN}$ decomposition (Iwasawa decomposition), i.e. let $g \in$ Möb then $g = kan$, $k \in \mathbf{K}$, $a \in \mathbf{A}$, $n \in \mathbf{N}$. The subgroup $\mathbf{A}$ preserves the upper semicircle, or the right half-line on the line picture, while $\mathbf{N}$ maps it into itself for $s > 0$. By the adjoint action of Möb one can define translation and dilation groups relative to any other interval I, resp. τ_I and δ_I.

Another convenient choice replaces the rotations by the special conformal transformations $I \circ t \circ I$, where the conformal inversion $I : t \mapsto -1/t$ in the line picture is the rotation by π in the circle picture.

Unitary positive-energy representations of $\widetilde{\text{Möb}}$. Let U be a unitary representation of $\widetilde{\text{Möb}}$ on a Hilbert space $\mathcal{H}$. The self-adjoint infinitesimal generator of the rotation subgroup in U is denoted by L_0, i.e. $U(r(\theta)) = e^{i\theta L_0}$. L_0 is called the *conformal Hamiltonian*. Let P, D, K be the generators of the translations, dilations and special transformations, resp., and (by abuse of notation) I the unitary representative of the conformal inversion, then one has

$$IPI = K, \quad IDI = -D, \quad L_0 = \frac{1}{2}(P + K), \quad I = e^{\pi L_0}. \tag{2.2}$$

U is said to be a **positive-energy representation of $\widetilde{\text{Möb}}$** if the spectrum of the conformal Hamiltonian L_0 is contained in $[0, +\infty)$.

Irreducible, unitary, positive energy representations of $\widetilde{\text{Möb}}$ on a Hilbert space $\mathcal{H}$ are labelled by positive real numbers k. They correspond to the lowest eigenvalue of the conformal Hamiltonian L_0, called "lowest weight". An irreducible positive unitary representation of $\widetilde{\text{Möb}}$ factors on Möb, iff k is an integer.

Let $\mathbf{P}$ be the translation-dilation subgroup of Möb associated to $\mathbb{R}^+$. A unitary representation of $\mathbf{P}$ is said to have **positive energy** if the spectrum of the translation subgroup is contained in the positive half-line $[0, +\infty)$. There exists a unique, up to unitary equivalence, irreducible unitary positive-energy representation U of $\mathbf{P}$. The positivity of the energy of a $\widetilde{\text{Möb}}$ representation U is equivalent to the positivity of the translation generator, thus a positive-energy representation U of Möb restricts to the unique positive-energy representation of $\mathbf{P}$ [12]. Furthermore, if U is irreducible then $U|_{\mathbf{P}}$ is irreducible [22].

2.4 (Anti-)unitary extensions

The Poincaré group. Let θ be the space and time reflection $(t, \vec{x}) \mapsto (-t, -\vec{x})$ and α be the action of θ on $\mathcal{P}_+^\uparrow$ by conjugation, we define

$$\mathcal{P}_+ = \mathbb{Z}_2 \ltimes_\alpha \mathcal{P}_+^\uparrow$$

to be the extension of $\mathcal{P}_+^\uparrow$ through α. An (anti-)unitary representation of $\mathcal{P}_+$ is unitary, resp. anti-unitary, on $\mathcal{P}_+^\uparrow$ resp. on $\theta \mathcal{P}_+^\uparrow$.

Proposition 2.1. *[29] A unitary irreducible positive energy representation U of $\mathcal{P}_+^\uparrow$ extends (anti-)unitarily to $\mathcal{P}_+$ iff it is induced by a self-conjugate representation of the little group.*

This is true for all irreducible positive energy representations except for those of non-zero finite helicity. On the other hand $U_h \oplus U_{-h}$ extends to $\mathcal{P}_+$.

The Möb$_2$ group. Let r be the complex conjugation $z \mapsto \overline{z}$ on S^1 ($x \mapsto -x$ in $\overline{\mathbb{R}}$), and α be the action of r on Möb by conjugation, we define

$$\text{Möb}_2 = \mathbb{Z}_2 \ltimes_\alpha \text{Möb}$$

to be the extension of Möb through α. Note that r reverses the orientation. An (anti-)unitary representation of Möb$_2$ is unitary, resp. anti-unitary, on Möb (the orientation preserving transformations of Möb$_2$) resp. on rMöb (the orientation reversing transformations of Möb$_2$).

Proposition 2.2. *[22] Every unitary positive energy representation U of Möb extends (anti-)unitarily to Möb$_2$.*

Now we are going to show that there exists a unique, up to unitary equivalence, way to represent (anti-)unitarily such extensions. Let K be a locally compact group, α be an involutive automorphism of K and G be the semidirect product $K \rtimes_\alpha \mathbb{Z}_2$. Let U and $\hat{U} \doteq U \circ \alpha$ be unitary representations of K on a Hilbert space $\mathcal{H}$ and J be an anti-unitary operator on $\mathcal{H}$, we shall call $J\hat{U}J^*$ the *conjugate representation* of U. The unitary equivalence class of the conjugate representation does not depend on the choice of J. If $\alpha = 1$, then our definition of conjugate representation coincides with the classical one. An (anti-)unitary representation of G is unitary on K and anti-unitary on rK, where r is the $\mathbb{Z}_2$-generator. U is said to be *self-conjugate* if U is unitary equivalent to $J\hat{U}J^*$, and *real* if the anti-unitary J can be chosen s.t. $J^2 = 1$ and

$$U = J\hat{U}J^*. \tag{2.3}$$

Note that such an anti-unitary involution extends the representation U (anti-)unitarily from K to G (the converse is also true). In case J can be chosen s.t. $J^2 = -1$ and (2.3) holds, then U is said to be *pseudo-real*.

Proposition 2.3. *Assume that K is a locally compact type I group, and let U be a unitary representation of K. Then*

1. *If U is real, then it extends to an (anti-)unitary representation of G on $\mathcal{H}$. The extension is unique modulo unitary equivalence.*

2. *In general, let J be an anti-unitary involution on $\mathcal{H}$, then $U \oplus J\hat{U}J$ is real and it extends uniquely (up to unitary equivalence) to an (anti-)unitary representation of G on $\mathcal{H} \oplus \mathcal{H}$ as a consequence of point 1.*

Proof. Firstly, we consider the factorial case, namely $U = U_0 \otimes \mathbf{1}$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{K}$, where U_0 is an irreducible unitary representation of K. We consider the following cases:

(1.a) Assume that U and U_0 are real representations. In this case, U_0 extends to an (anti-)unitary representation of G through an anti-unitary operator J_0 satisfying (2.3). Let $\mathbf{J}$ be any antilinear involution on $\mathcal{K}$, one can define the anti-unitary involution $J = J_0 \otimes \mathbf{J}$, which extends U (anti-)unitarily to G and satisfies (2.3).

Now we have to show the uniqueness up to unitary equivalence of the extension. Consider another anti-unitary involution J' on $\mathcal{H}$ extending U from K to G. The composition $JJ' \in U(K)' = \mathbb{C} \otimes \mathcal{B}(\mathcal{K})$ and since JJ' is a unitary,

$$J' = (1 \otimes Z)(J_0 \otimes \mathbf{J}) = J_0 \otimes Z\mathbf{J}$$

where Z is a unitary operator on the Hilbert space $\mathcal{K}$. By uniqueness, up to unitary equivalence, of the complex structure of an Hilbert space, then there exists a unitary $V \in \mathcal{U}(\mathcal{K})$ s.t. $V\mathbf{J}V^* = Z\mathbf{J}$, thus

$$(1 \otimes V)J(1 \otimes V^*) = J'.$$

The two extensions through J and J' are unitarily intertwined by $1 \otimes V$.

(1.b) Assume that U is real and U_0 is pseudo-real, w.r.t. an anti-unitary operator J_0. Let $\mathbf{J}$ be an anti-unitary operator s.t. $\mathbf{J}^2 = -1$ then $J = J_0 \otimes \mathbf{J}$ is an involution implementing the $\mathbb{Z}_2$-generator on U and satisfying (2.3). The argument of unitary equivalence of the (anti-)unitary extensions is a slight modification of the previous case.

(1.c) Assume that U_0 is disjoint from the conjugate representation. Let J_0 be an antilinear involution on $\mathcal{H}_0$, we define representation

$$\tilde{U} = \left(U_0 \oplus J_0 \hat{U}_0 J_0 \right) \otimes \mathbf{1}_{\mathcal{K}}$$

acting on $\tilde{\mathcal{H}} = (\mathcal{H}_0 \oplus \mathcal{H}_0) \otimes \mathcal{K}$. $\tilde{U}$ is real w.r.t. the following anti-unitary involution $\tilde{J}$. Let σ be the flip operator on $\mathcal{H}_0 \oplus \mathcal{H}_0$, i.e., $\sigma(\xi \oplus \eta) = \eta \oplus \xi$, $\mathbf{J}$ be an antilinear involution on $\mathcal{K}$, then we define

$$\tilde{J} = ((J_0 \oplus J_0) \cdot \sigma) \otimes \mathbf{J}$$

on $\tilde{\mathcal{H}}$. It extends (anti-)unitarily $\tilde{U}$ from K to G. Let $\tilde{J}'$ be another anti-unitary involution extending $\tilde{U}$ to G, then $\tilde{J}\tilde{J}' \in U(K)' = (\mathbb{C} \oplus \mathbb{C}) \otimes B(\mathcal{K})$. Since $\tilde{J}^2 = 1$, it is easy to see that there exists a unitary $V \in \mathcal{U}(\mathcal{K})$ such that

$$\left(\mathbf{1}_{\mathbb{C}^2} \otimes V \right) \tilde{J} \left(\mathbf{1}_{\mathbb{C}^2} \otimes V^* \right) = \tilde{J}'$$

by uniqueness of the complex structure of the Hilbert space $\mathcal{K}$, and we conclude this case.

We sketch the proof for the general case. Since K is a type I group the above result generalizes to direct integrals and direct sums of factorial representations. Indeed, for $U = \int_X U_x d\mu(x)$ where $\{U_x\}_{x \in X}$ is a family of factorial representations and (X, μ) is a standard measure space, the product of any two anti-unitary involutions J and J' extending U to G belongs to $U(K)'$. Then one can conclude by applying the factorial case on integral fibers. $\square$

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Positive-energy unitary representations of $\mathcal{P}_+^\uparrow$ and Möb satisfy the assumptions of Proposition 2.3. In particular, positive-energy factorial representations of Möb fall in case (1.a); massive, scalar massless and infinite spin $\mathcal{P}_+^\uparrow$-representations also fall in (1.a), and massless non-zero helicity representations in (1.c).

3 One-particle nets and Brunetti-Guido-Longo construction

In QFT, localization is formulated in terms of *local nets*, i.e., inclusion preserving maps that associate with open spacetime regions the corresponding quantum structures (algebras or Hilbert spaces, see below), and Einstein causality is encoded as a feature of these maps. We introduce the various nets pertaining to our purpose.

The general idea of the connection between nets of algebras on a complex Hilbert space containing the vacuum vector Ω and nets of real Hilbert subspaces is to define, for every spacetime region O, $H(O) := \overline{\mathcal{A}(O)_{sa}\Omega}$.[2] One may also take the intersection with the one-particle space $H_1(O) = H(O) \cap \mathcal{H}_1$. In the free case, one can recover $H(O)$ and also $\mathcal{A}(O)$ from the real Hilbert spaces $H_1(O)$ by second quantization, and this can be used as a construction, once $H_1(O)$ are given. Finally, modular theory allows to define $H_1(O)$ intrinsically in terms of a positive-energy representation of $\mathcal{P}_+^\uparrow$.

3.1 Standard subspaces

A linear, real, closed subspace H of a complex Hilbert space $\mathcal{H}$ is called **cyclic** if $H + iH$ is dense in $\mathcal{H}$, **separating** if $H \cap iH = \{0\}$ and **standard** if it is cyclic and separating.

Given a standard subspace H the associated **Tomita operator** S_H is defined to be the closed the anti-linear involution with domain $H + iH$, given by:

$$S_H : H + iH \ni \xi + i\eta \mapsto \xi - i\eta \in H + iH, \qquad \xi, \eta \in H,$$

on the dense domain $H + iH \subset \mathcal{H}$. The polar decomposition

$$S_H = J_H \Delta_H^{1/2}$$

defines the positive self-adjoint **modular operator** Δ_H and the anti-unitary **modular conjugation** J_H. In particular, Δ_H is invertible and

$$J_H \Delta_H J_H = \Delta_H^{-1}.$$

If H is a real linear subspace of $\mathcal{H}$, the *symplectic complement* of H is defined by

$$H' \equiv \{\xi \in \mathcal{H} \; ; \; \Im(\xi, \eta) = 0, \forall \eta \in H\} = (iH)^{\perp_\mathbb{R}} \; ,$$

where $\perp_\mathbb{R}$ denotes the orthogonal in $\mathcal{H}$ viewed as a real Hilbert space with respect to the real part of the scalar product on $\mathcal{H}$. H' is a closed, real linear subspace of $\mathcal{H}$. If H is standard, then $H = H''$. It is a fact that H is cyclic (resp. separating) iff H' is separating (resp. cyclic), thus H is standard iff H' is standard and we have

$$S_{H'} = S_H^* \; .$$

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[2] M_{sa} are the self-adjoint elements of a von Neumann algebra M.

Fundamental properties of the modular operator and conjugation are

$$\Delta_H^{it} H = H, \quad J_H H = H' , \qquad t \in \mathbb{R} .$$

The one-parameter, strongly continuous group $t \mapsto \Delta_H^{it}$ is the **modular group** of H (cf. [28]).

There is a 1–1 correspondence between Tomita operators and standard subspaces, namely between:

- Standard subspaces $H \subset \mathcal{H}$,

- Closed, densely defined anti-linear involutions S on $\mathcal{H}$,

- Pairs (J, Δ) of an anti-unitary involution J and a positive self-adjoint operator Δ on $\mathcal{H}$ s.t.

$$J\Delta J = \Delta^{-1}. \tag{3.4}$$

Namely, given (J, Δ) one can recover $S := J\Delta^{\frac{1}{2}}$ and H as the real eigenspace of S with eigenvalue 1.

We shall need the following results on standard subspaces.

Lemma 3.1. [22, 23]. *Let $H \subset \mathcal{H}$ be a standard subspace, and $K \subset H$ a closed, real linear subspace of K.*

If $\Delta_H^{it} K = K$, $\forall t \in \mathbb{R}$, then K is a standard subspace of $\mathcal{K} \equiv \overline{K + iK}$ and $\Delta_H|_K$ is the modular operator of K on $\mathcal{K}$. If moreover K is a cyclic subspace of $\mathcal{H}$, then $H = K$.

Lemma 3.2. [22, 23]. *Let $H \subset \mathcal{H}$ be a standard subspace, and U a unitary on $\mathcal{H}$ such that $UH = H$. Then U commutes with Δ_H and J_H.*

The following is the one-particle analogue of Borchers' theorem [4].

Theorem 3.3. [22, 23]. *Let $H \subset \mathcal{H}$ be a standard subspace, and U a one-parameter unitary group on $\mathcal{H}$ with positive generator, such that $U(t)H \subset H$, $t \geq 0$. Then $\Delta_H^{is} U(t) \Delta_H^{-is} = U(e^{-2\pi s}t)$.*

3.2   Nets on Minkowski spacetime

Let U be a unitary positive energy representation of the Poincaré group on a Hilbert space $\mathcal{H}$.

A U-covariant *net of standard subspaces* $\mathcal{H}$ on the set $\mathcal{W}$ of wedge regions of the Minkowski spacetime is a map

$$H : \mathcal{W} \ni W \longmapsto H(W) \subset \mathcal{H}$$

that associates a closed real linear subspace $H(W)$ with each $W \in \mathcal{W}$, satisfying:

1. *Isotony*: If $W_1 \subset W_2$ then $H(W_1) \subset H(W_2)$;

2. *Poincaré covariance*: $U(g)H(W) = H(gW)$ $(W \in \mathcal{W}, g \in \mathcal{P}_+^\uparrow)$;

3. *Reeh-Schlieder property*: $H(W)$ is cyclic $\forall\, W \in \mathcal{W}$;

4. *Locality*: For every wedge $W \in \mathcal{W}$ we have

$$H(W') \subset H(W)'.$$

The net is said to have the Bisognano-Wichmann property if

5. *Bisognano-Wichmann property*: $\Delta^{it}_{H(W)} = U\big(\Lambda_W(-2\pi t)\big)$ for all $W \in \mathcal{W}$ and $t \in \mathbb{R}$.

Given a U-covariant net H on $\mathcal{W}$, one gets a net of closed, real linear subspaces on double cones O defined by

$$H(O) \equiv \bigcap_{\mathcal{W} \ni W \supset O} H(W) . \tag{3.5}$$

Note that $H(O)$ is not necessarily cyclic. If $H(O)$ is cyclic and H has the BW property, then

$$H(W) = \overline{\sum_{O \subset W} H(O)}$$

by Lemma 3.1.

3.3 Nets on the circle

Let $\mathcal{I}$ be the set of nonempty, nondense, open connected intervals of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let U be a positive-energy representation of Möb on a Hilbert space $\mathcal{H}$.

A **Möbius covariant net** is a map H which assigns to every interval $I \in \mathcal{I}$ a von Neumann algebra $H(I) \subset \mathcal{H}$ satisfying the following properties:

1. *Isotony:* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \subset I_2$, then $H(I_1) \subset H(I_2)$;
2. *Möbius covariance:* $U(g)H(I) = H(gI)$ $(I \in \mathcal{I}, g \in \text{Möb})$;
3. *Reeh-Schlieder property:* $H(I)$ is cyclic for every $I \in \mathcal{I}$;
4. *Locality:* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \varnothing$,

$$H(I_1) \subset H(I_2)'.$$

With these properties, $H(I)$ are standard subspaces, and the net automatically satisfies the Bisognano-Wichmann property

5. *Bisognano-Wichmann property*: $\Delta^{it}_{H(I)} = U\big(\delta_I(-2\pi t)\big)$ for all $I \in \mathcal{I}$ and $t \in \mathbb{R}$.

Here, for I_+ the upper semicircle, $\delta_{I_+}(t)$ are the dilations $x \mapsto e^t x$ in the line picture, and for every other interval $I = g(I_+)$ $(g \in \text{Möb})$, $\delta_I(t) = g \circ \delta_{I_+}(t) \circ g^{-1}$.

A translation-dilation covariant net of standard subspaces on the intervals of the real line $\mathbb{R}$ can be defined in complete analogy. It is said to satisfy the Bisognano-Wichmann property if $U(\delta_{\mathbb{R}_+}(2\pi t)) = \Delta^{-it}_{\mathbb{R}_+}$. In this case, it is possible to obtain a net on the circle; and also the converse is true:

Lemma 3.4. *[22] Let H be a translation-dilation net on the line. It extends to a Möbius covariant net on the circle if and only if the Bisognano-Wichmann property holds. The extension is unique.*

3.4 Brunetti-Guido-Longo construction

The Brunetti-Guido-Longo construction relies on the 1–1 correspondence between standard subspaces and Tomita-Takesaki modular data.

On Minkowski space. [9] Let U be an (anti-)unitary representation of $\mathcal{P}_+$, and J_W and K_W the anti-unitary reflection and the self-adjoint generator of the one-parameter group of boosts associated with the wedge W (i.e., $U(\Lambda_W(t)) = e^{itK_W}$), respectively. The pair $(J_W, \Delta_W \equiv e^{-2\pi K_W})$, satisfies (3.4), thus one can associate to any wedge $W \in \mathcal{W}$ a standard subspace $H(W)$ in a covariant way. By positivity of the energy and the Borchers Theorem 3.3, the net $W \mapsto H(W)$ satisfies Isotony. Covariance, Locality and the Reeh-Schlieder and Bisognano-Wichmann properties hold by construction.

On the circle. [22] Let U be an (anti-)unitary representation of Möb and J_I and K_I the anti-unitary reflection and the generator of the one-parameter group of dilations associated with the interval I, respectively, and $\Delta_I = e^{-2\pi K_I}$. In analogy with the previous, we can define a net $I \mapsto H(I)$. By positivity of the energy and the Borchers theorem, the net $I \mapsto H(I)$ satisfies Isotony. Covariance, Locality and the Reeh-Schlieder and Bisognano-Wichmann properties hold by construction.

Proposition 3.5. *There is a unique, up to unitary equivalence, net of standard subspaces on the considered spacetimes (circle or Minkowski) satisfying 1.–5. of Sect. 3.2 resp. 3.3.*

Proof. *On Minkowski space.* Let $W \mapsto H(W)$ be a Poincaré covariant net of standard subspaces on wedges satisfying the Bisognano-Wichmann property w.r.t. a positive-energy unitary Poincaré representation U. Then the modular conjugations of wedge subspaces extend U to an (anti-)unitary representation of $\mathcal{P}_+$, cf. [14]. We conclude by the unitary equivalence of (anti-)unitary extensions in Proposition 2.3 and the 1–1 correspondence between Tomita operator and standard subspaces.

On the circle. Let $I \mapsto H(I)$ be a Möbius covariant net of standard subspaces on intervals satisfying 1.–5. Then U extends to an (anti-)unitary representation of Möb$_2$ through interval modular conjugations [22]. The conclusion again follows by Proposition 2.3 and the 1–1 correspondence between Tomita operators and standard subspaces. $\qquad\square$

3.5 Chiral current models [12]

For $n \in \mathbb{N}$ consider the Hilbert space $\mathcal{H}_n$ defined by the closure of the space of square integrable functions on $\overline{\mathbb{R}}$ w.r.t. the scalar product

$$(f,g)_n = \int_0^\infty p^{2n-1} dp \, \overline{\widehat{f}(p)} \widehat{g}(p).$$

Its null space contains the polynomials of degree $2(n-1)$. The associated symplectic form on the real-valued functions is

$$\omega_n(f,g) \equiv \Im(f,g)_n = \frac{(-1)^{n-1}}{2} \int f(x)g(y)\delta^{(2n-1)}(x-y) \, dx \, dy.$$

$\mathcal{H}_n$ carries a unitary positive-energy representation U_n of Möb by

$$(U_n(g)f)(x) = \left(\frac{dg(x)}{dx}\right)^{-2(n-1)} f(g(x)) = (cx - a)^{2(n-1)} f(g(x)), \tag{3.6}$$

in particular

$$(If)(x) = x^{2(n-1)} \cdot f(I(x)). \tag{3.7}$$

The self-adjoint generators act by

$$(Pf)(x) = i\partial_x f(x), \quad (Df)(x) = i(x\partial_x - (n-1))f(x), \tag{3.8}$$

$$(Kf)(x) = i(x^2\partial_x - 2(n-1)x)f(x). \tag{3.9}$$

U_n is the positive-energy representation of lowest weight n.

Applying the BGL construction (Sect. 3.4) to the representation U_n, one obtains the net of real subspaces

$$I \mapsto H_n(I) = \overline{\{f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) : \mathrm{supp}\, f \subset I\}}^{\|\cdot\|_n} \subset \mathcal{H}_n,$$

on which Möb acts covariantly. ("Modular localization" is the fact that the support property arises as a consequence of the definition of $H_n(I)$ via modular theory. Locality is then seen directly from the symplectic form ω_n.) The second quantization of H_n (cf. Sect. 3.6) gives the net of von Neumann algebras generated by the quasiprimary chiral current j_n of dimension n.

Note that

$$(f,g)_n = (\partial_x f, \partial_x g)_{n-1},$$

i.e., the derivative ∂_x is a unitary operator $\mathcal{H}_n \to \mathcal{H}_{n-1}$. This operator intertwines the actions of the generators P and D, but not of K. Thus, $\partial_x : \mathcal{H}_n \to \mathcal{H}_{n-1}$ implements the unitary equivalence of the restrictions of U_n to the translation-dilation subgroup, cf. Sect. 2.3. In the language of quantum field theory, this is the statement that a quasiprimary current of dimension n and the derivative of a quasiprimary current of dimension $n-1$ transform in the same way under translations and dilations.

The distinction is only seen by the action of K, or I. This motivates the following

Lemma 3.6. *Suppose a representation U of Möb is given whose restriction to $\mathbf{P}$ is given by (3.8). Suppose that I acts geometrically, i.e.,*

$$(If)(x) = g(x)f(I(x)) \tag{3.10}$$

with some function g. Then $g(x) = x^{2(n-1)}$, and $U = U_n$.

Proof. By direct computations, using (2.2): Insertion of (3.10) into $ID + DI = 0$ implies that g is homogeneous of degree $2(n-1)$, hence $g(x) = g_0 x^{2(n-1)}$. $I^2 = \mathrm{id}$ implies $g_0 = \pm 1$. Then $K = IPI$ implies (3.9), which together with (3.8) integrates to (3.6). Then (3.7) implies $g_0 = 1$. $\square$

The following is a reformulation of Lemma 3.6.

Proposition 3.7. *Let $\mathcal{H}$ be the anti-Fourier transform of the Hilbert space $L^2(\mathbb{R}, p^{2k+1}dp)$. Let*

$$I \mapsto H(I) = \overline{\{f \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}), \operatorname{supp} f \subset I\}} \subset \mathcal{H}$$

be a Möb-covariant standard subspace net with the natural action of translations and dilations on $\mathcal{H}$. Then $H(I) = \overline{\{j_k(f)\Omega \in \mathcal{H} : \operatorname{supp} f \subset I\}}$ where j_k is the quasiprimary field of dimension k. In particular, H is the one-particle net H_k associated to U^k (up to multiplicity).

Proof. By the Bisognano-Wichmann property we can say H is the canonical BGL net associated to the covariant Möb-representation. Thus there exists a current j generating H.

Assume that, the scalar product is misidentified, for simplicity consider $(f, f) = \|j'_{k-1}(f)\Omega\|^2$. It is possible since j_k and $\partial_x j_{k-1}$ share the same scale dimension, and the same translation-dilation covariant representation (but inequivalent Möb covariant representations). Then the conformal inversion would act geometrically on derivatives as f' because $j'_{h-1}(f) = -j_{h-1}(f')$, but not on its primitive f. As a consequence $j = j_k$ and $H = H_k$. $\qquad\square$

3.6 Second quantization and nets of von Neumann algebras

With $\mathcal{H}$ a Hilbert space and $H \subset \mathcal{H}$ a real linear subspace, $R_+(H)$ is the von Neumann algebra on the symmetric Fock space $F_+(\mathcal{H})$ generated by the CCR operators:

$$R_+(H) \equiv \{w(f) : f \in H\}'', \tag{3.11}$$

with $w(\xi)$ the Weyl unitaries on $F_+(\mathcal{H})$ defined on the coherent states $e^g \in F_+(\mathcal{H})$ ($f \in \mathcal{H}$) by their action $w(f)e^g = e^{-\frac{1}{2}(f,f)-(f,g)} \cdot e^{f+g}$. If $\varphi(f)$ is the selfadjoint generator of the unitary one-parameter group $w(f)$, this standard construction ensures the identification of the "one-particle vector" $\varphi(f)\Omega \in F_+(\mathcal{H})$ with $f \in \mathcal{H} \subset F_+(\mathcal{H})$. By continuity we have that

$$R_+(H) = R_+(\overline{H}) \ .$$

Moreover the Fock vacuum vector Ω is cyclic (resp. separating) for $R_+(H)$ iff $\overline{H}$ is cyclic (resp. separating).

Second quantization respects the lattice structure [1] and the modular structure [21, 24]. We recall these basic properties. For a standard subspace $H \subset \mathcal{H}$, we denote by S_H^+, J_H^+, Δ_H^+ the Tomita operators associated with $(R_+(H), \Omega)$, and by $\Gamma_+(T)$ the Bose second quantization of a one-particle operator T on $\mathcal{H}$.

Proposition 3.8. *[1, 21, 24] Let H and H_a be closed, real linear subspaces of $\mathcal{H}$. We have*

(a) $R_+(H)' = R_+(H')$;

(b) $R_+(\sum_a H_a) = \bigvee_a R_+(H_a)$;

(c) $R_+(\bigcap_a H_a) = \bigcap_a R_+(H_a)$.

(d) *If H is standard, then $S_H^+ = \Gamma_+(S_H)$, $J_H^+ = \Gamma_+(J_H)$, $\Delta_H^+ = \Gamma_+(\Delta_H)$.*

Given the canonical BGL-net H_U associated with a unitary positive-energy representation U of $\mathcal{P}_+^\uparrow$ or Möb, then its second quantization net

$$\mathcal{A}(W) \equiv R_+\big(H_U(W)\big)\,, \quad W \in \mathcal{W}\,, \quad \text{resp.} \quad \mathcal{A}_n(I) \equiv R_+\big(H_{U_n}(I)\big)\,, \quad I \in \mathcal{I}\,,$$

is the free field net, i.e., $\mathcal{A}(W)$ is generated by Weyl operators $\mathrm{w}(f) = e^{i\varphi(f)}$ of free Wightman fields smeared with real test functions supported in W, and $\mathcal{A}_n(I)$ is generated by Weyl operators $\mathrm{w}(f) = e^{ij_n(f)}$ of the quasiprimary current of dimension n, smeared in I. The case $n = 1$ is the canonical $U(1)$ current.

These nets satisfy the usual assumptions on nets of von Neumann algebras of local observables.

On Minkowski space.

- *Isotony*: $\mathcal{A}(W_1) \subset \mathcal{A}(W_2)$ if $W_1 \subset W_2$;
- *Poincaré covariance*: U is a positive-energy representation of $\mathcal{P}_+^\uparrow$, and $U(g)\mathcal{A}(W)U(g)^* = \mathcal{A}(gW)$, $g \in \mathcal{P}_+^\uparrow$;
- *Vacuum with Reeh-Schlieder property*: there exists a unique (up to a phase) U-invariant vector $\Omega \in \mathcal{H}$, and Ω is cyclic and separating for $\mathcal{A}(W)$ for all $W \in \mathcal{W}$;
- *Locality*: $\mathcal{A}(W') \subset \mathcal{A}(W)'$.

In addition, for the canonical free field nets the *Bisognano-Wichmann property* holds:

$$\Delta_{\mathcal{A}(W),\Omega}^{it} = U\big(\Lambda_W(-2\pi t)\big), \quad W \in \mathcal{W},\ t \in \mathbb{R}\,,$$

where $\Delta_{\mathcal{A}(W),\Omega}$ is the modular operator of $(\mathcal{A}(W),\Omega)$.

On the circle.

- *Isotony*: $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ if $I_1 \subset I_2$;
- *Möbius covariance*: U is a positive-energy representation of Mob, and $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$, $g \in$ Möb;
- *Vacuum*: There exists a unique (up to a phase) U-invariant vector $\Omega \in \mathcal{H}$;
- *Locality*: $\mathcal{A}(I') \subset \mathcal{A}(I)'$, $I \in \mathcal{I}$;

The following are consequences of these axioms

- *Reeh-Schlieder property*: Ω is a cyclic and separating vector for each $\mathcal{A}(I)$, $I \in \mathcal{I}$;
- *Haag duality*: $\mathcal{A}(I')' = \mathcal{A}(I)$, $I \in \mathcal{I}$;
- *Bisognano-Wichmann property*: $U(\delta_I(-2\pi t)) = \Delta_{\mathcal{A}(I),\Omega}^{it}$, for all $I \in \mathcal{I}$ and $t \in \mathbb{R}$.

4   Time-axis theory of finite helicity representations

Consider the representation $U = U_h \oplus U_{-h}$ of the Poincaré group. The Brunetti-Guido-Longo construction associates to U a net of standard subspaces H on wedge shaped regions satisfying the Bisognano-Wichmann property. The second quantization procedure provides the free field net $\mathcal{A}$ associated with U.

Finite helicity von Neumann algebra nets have an associated Wightman field ϕ_h satisfying the Bisognano-Wichmann property [2]. Thus the BGL and the Wightman field constructions coincide as

$$H(O) = \overline{\{\phi_h(f)\Omega : f \in \mathcal{C}_0^\infty(\mathbb{R}^{1+3}),\ \mathrm{Supp}f \subset O\}} \quad \text{and} \quad H(W) = \bigcup_{O \subset W} H(O),$$

gives a one-particle U-covariant net (with two polarizations h and $-h$) and its second quantization

$$\mathcal{A}_h(O) \doteq R_+(H(O)) = \{e^{i\phi_h(f)} : f \in \mathcal{C}_0^\infty(\mathbb{R}^{1+3}),\ \mathrm{Supp}f \subset O\}\}''$$

gives the free field. Note that Haag duality holds by [17, 16], namely $H(O') = H(O)'$ and $R_+(H(O')) = R_+(H(O)') = R_+(H(O))'$. Furthermore, due to the conformal covariance, the modular operator of any double cone subspace (resp. second quantization algebra) implement a one-parameter group of conformal transformation that is conjugated to the dilation and the boost one parameter groups [16].

Firstly, note that it is not possible to unitarily rewrite the net H as a direct sum according to $U_h \oplus U_{-h}$, as U_h does not extend (anti-)unitarily to $\mathcal{P}_+$ [19, 29]. On the other hand $U_{\pm h}$ (thus U) extends to a representation $\widetilde{U}_{\pm h}$ (resp. $\widetilde{U}$) of the conformal group which acts covariantly on the net H, see e.g. [17, 16, 18].

We recall that a local net of standard subspaces on double cones undergoing the action of a massless Poincaré representation is timelike local.

Lemma 4.1. *[24] Assume that U is a massless, unitary representation of $\widetilde{\mathcal{P}}_+^\uparrow$ acting covariantly on a local net of closed, real linear subspaces on double cones. Let O_1, O_2 be double cones with O_2 in the time-like complement of O_1, then*

$$H(O_2) \subset H(O_1)' \ ,$$

where $H(O) = \bigcap_{W \supset O} H(W)$.

Now, we can define a local net of standard subspaces on the time axis. Let $I = (a, b) \subset \mathbb{R}$ be an interval and $O_I = (V_- + b) \cap (V_+ + a)$ the double cone with vertices on the time axis. Then we get a net on the line

$$I \mapsto H(I) = H(O_I)$$

which undergoes the Möbius covariant action of $\widetilde{U}|_{\mathsf{M\ddot{o}b}}$.

Since any unitary positive energy Möbius representation extends (anti-)unitarily to $\mathsf{M\ddot{o}b}_2$, then $\widetilde{U}_{\pm h}|_{\mathsf{M\ddot{o}b}}$ extends to $\mathsf{M\ddot{o}b}_2$ and acts covariantly on its BGL-standard subspace net H_0. By Proposition 3.5 and the Bisognano-Wichmann property for the dilation group, we have that the net $I \mapsto H(I)$ is unitarily equivalent to the direct sum of the two local $\mathsf{M\ddot{o}b}$-covariant nets $H_\pm$.

Now we need the structure coming from Wightman fields in order to construct the theories on the time axis.

4.1   One-particle space and free field equations

Free field theories are completely determined by their one-particle structure. This structure is conveniently described by the two-point functions of Wightman fields, that define the one-particle space by endowing the space of test functions with a scalar product. The null

space of this scalar product is completely characterized by the free field equations (that are closer to the physicists' mind). Our strategy is to use the latter in order to control the one-particle space and the pertinent decomposition of the one-particle representations.

As compared to the scalar field, there are two complications with helicity > 1: the (higher) Maxwell fields transform nontrivially under SO(3), and the one-particle space of a local field carries necessarily the direct sum of the irreducible representations of helicity $+h$ and $-h$. (Nevertheless, we shall loosely refer to the local fields as "helicity-h fields".)

The electromagnetic field, $h = 1$. The Maxwell equations for the magnetic and electric fields in absence of charges are

$$\operatorname{curl} \mathbf{B} = \frac{\partial}{\partial t}\mathbf{E}, \qquad \operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \qquad \operatorname{div} \mathbf{E} = 0 = \operatorname{div} \mathbf{B}.$$

The field strength $F_{\mu\nu}$ of the electromagnetic field is defined to be the anti-symmetric tensor given by

$$\mathbf{E} = (F_{01}, F_{02}, F_{03}) \quad \text{and} \quad \mathbf{B} = (F_{32}, F_{13}, F_{21}),$$

and the Maxwell equations become

$$\partial^{\mu}F_{\mu\nu} = 0 \quad \text{and} \quad \partial_{\mu}F_{\nu\rho} + \partial_{\nu}F_{\rho\mu} + \partial_{\rho}F_{\mu\nu} = 0.$$

These imply the Klein-Gordon equation $\Box F = 0$.

In the quantized theory, the one-particle Hilbert space is the space of test functions $f^{\mu\nu}$ equipped with the scalar product given by the two-point function

$$(f, f) := (F(f)\Omega, F(f)\Omega) = (\Omega, F(\overline{f})F(f)\Omega).$$

The latter is dictated by covariance (i.e., by Weinberg's quantization [30] based on Wigner's intrinsic construction [31] avoiding the use of a potential) to be

$$\|F(f)\Omega\|^2 = \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{|\mathbf{p}|} \, p_\mu \, p_\tau \, \eta_{\nu\sigma} \, \overline{\hat{f}^{\mu\nu}}(p) \, \hat{f}^{\sigma\tau}(p).$$

Higher correlations are obtained by Wick's theorem, so that the full Hilbert space is the Fock space, and multi-particle states can be created by the usual creation and annilation operators. The field strength transforms covariantly under the Poincaré group:

$$U(a, \Lambda)F_{\mu\nu}(x)U(a, \Lambda)^* = \Lambda_\mu^{\rho} \, \Lambda_\nu^{\sigma} F_{\rho\sigma}(\Lambda x + a).$$

It is well known that $U(a, \Lambda)$ acts on the one-particle space as the direct sum of Poincaré representations of helicities 1 and -1 [30].

In order to prove the split property for the resulting net, we want to restrict the Maxwell net to the time axis. This gives a chiral conformal QFT. By computing $\operatorname{Tr} e^{-\beta L_0}$ for this chiral QFT and showing that it is finite for all $\beta > 0$, we shall establish that the chiral net satisfies the split property. From this, we can conclude that the original net has the split property.

Before we present the purely representation-theoretical argument for arbitrary helicities $|h| \geq 1$, we want to give its field-theoretic version in the Maxwell case.

The Poincaré transformations of the Maxwell tensor extend to the conformal group by

$$U(g)F_{\mu\nu}(x)U(g)^* = J_g(x)_\mu^{\rho} J_g(x)_\nu^{\sigma} F_{\rho\sigma}(g(x)),$$

where $J_g(x)_\mu^\rho = \partial g(x)^\rho / \partial x^\mu$ is the Jacobi matrix. For infinitesimal transformations with generators P_0 (time translations), D (dilations) and $K_0 = IP_0I$ (special conformal transformations), one finds

$$i[P_0, F_{\mu\nu}(x)] = \partial_0 F_{\mu\nu}(x), \quad i[D, F_{\mu\nu}(x)] = (x^\kappa \partial_\kappa + 2)F_{\mu\nu}(x),$$

$$i[K_0, F_{\mu\nu}(x)] = \big(2x_0(x\partial) - x^2\partial_0 + 4\,x_0\big)F_{\mu\nu}(x) + 2(\eta_{0\mu}x^\kappa F_{\kappa\nu} - x_\mu F_{0\nu} - (\mu \leftrightarrow \nu)).$$

From this, the commutators with the restricted fields $\nabla_{\underline{a}}F_{\mu\nu}(t) = \nabla_{a_1} \dots \nabla_{a_k} F_{\mu\nu}(t, \vec{x})|_{\vec{x}=0}$ can be explicitly worked out.

Now, $P_0 = P$, D and $K_0 = K$ are the generators of Möb, and quasi-primary chiral currents of dimension h transform as

$$i[P, j(t)] = \partial_t j(t), \quad i[D, j(t)] = (t\partial t + h)j(t), \quad i[K, j(t)] = (t^2\partial t + 2ht)j(t).$$

It is obvious that the first two equations are satisfied by $\nabla_{\underline{a}}F_{\mu\nu}(t)$ with $h = 2 + |\underline{a}| = 2 + k$; but the last one is in general not fulfilled. In an SO(3)-covariant formulation, and using the Maxwell equations, we can bring the commutator with K into the form

$$i[K, J_{\underline{a},b}(t)] - (t^2\partial_t + 2(2+k)t)J_{\underline{a},b}(t) = 2 \sum_{1 \le i < j \le k} \partial_t \delta_{a_i a_j} J_{\underline{\ddot{a}},b}(t) + 2i \sum_{1 \le i \le k} \varepsilon_{a_i bc} J_{\underline{\dot{a}},c}(t),$$

where $J_{\underline{a},b}$ are the complex fields $\nabla_{a_1} \dots \nabla_{a_k}(E_b(t, \vec{x}) + iB_b(t, \vec{x}))|_{\vec{x}=0}$, and $\underline{\dot{a}}$ is the multi-index with a_i deleted, and similarly $\underline{\ddot{a}}$ is the multi-index with a_i and a_j deleted.

The quasi-primary currents are those for which the right-hand side vanishes. It is easy to see that this is precisely the case for the completely symmetric and traceless part of the rank $k + 1$ tensor $J_{a_1 \dots a_k, b}$. This tensor carries the spin $s = k + 1$-representation of SO(3), and because J is complex, there are two $2s + 1$ multiplets of real quasi-primary currents of dimension (= lowest weight of L_0) $2 + k$. All other components of $J_{a_1 \dots a_k, b}$ can be seen to be time derivatives of lower currents by virtue of the Maxwell equations $\partial_a J_a = 0$, $\partial_a J_b - \partial_b J_a = i\varepsilon_{abc}\partial_t J_c$ and the wave equation that follows from them.

Now, it is well known that on the subspace generated from the vacuum by a quasi-primary field of dimension h, one has $\text{Tr}_h\, e^{-\beta L_0} = \frac{e^{-\beta h}}{1 - e^{-\beta}}$, hence on the one-partical space of the Maxwell field,

$$\text{Tr}\, e^{-\beta L_0} = 2 \sum_{k \ge 0} (2k + 3) \cdot \frac{e^{-\beta(2+k)}}{1 - e^{-\beta}},$$

which can be easily summed as a geometric series in $z = e^{-\beta}$ with radius of convergence 1.

Higher helicity fields, $h > 1$. The field strength is a tensor

$$F_{[\mu_1\nu_1]\dots[\mu_h\nu_h]},$$

anti-symmetric in each index pair $[\mu\nu]$. It transforms covariantly under the Poincaré group:

$$U(a, \Lambda)F_{[\mu_1\nu_1]\dots[\mu_h\nu_h]}(x)U(a, \Lambda)^* = \Lambda_{\mu_1}^{\rho_1} \dots \Lambda_{\mu_h}^{\rho_h} \Lambda_{\nu_1}^{\sigma_1} \dots \Lambda_{\nu_h}^{\sigma_h} F_{[\rho_1\sigma_1]\dots[\rho_h\sigma_h]}(\Lambda x + a)$$

and is subject to the linear dependencies (symmetries)

$$F_{\dots[\mu_j\nu_j]\dots[\mu_k\nu_k]\dots} = F_{\dots[\mu_k\nu_k]\dots[\mu_j\nu_j]\dots}, \quad \eta^{\mu_j\mu_k}F_{[\mu_1\nu_1]\dots[\mu_h\nu_h]} = 0,$$

$$F_{[\alpha\beta][\gamma\nu_2]\dots[\mu_h\nu_h]} + F_{[\beta\gamma][\alpha\nu_2]\dots[\mu_h\nu_h]} + F_{[\gamma\alpha][\beta\nu_2]\dots[\mu_h\nu_h]} = 0. \tag{4.12}$$

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Its equations of motion ("higher Maxwell equations") are

$$\partial^\alpha F_{[\alpha\nu_1]\ldots[\mu_h\nu_h]} = 0, \quad \partial_\alpha F_{[\beta\gamma]\ldots[\mu_h\nu_h]} + \partial_\beta F_{[\gamma\alpha]\ldots[\mu_h\nu_h]} + \partial_\gamma F_{[\alpha\beta]\ldots[\mu_h\nu_h]} = 0. \qquad (4.13)$$

One can solve the linear dependencies in an $SO(3)$-covariant way by introducing the "electric" and "magnetic" components

$$E_{b_1\ldots b_h} := F_{[0b_1]\ldots[0b_h]}, \quad B_{b_1\ldots b_h} := \varepsilon_{b_1 j_1 k_1} F_{[j_1 k_1][0b_2]\ldots[0b_h]},$$

so that both E and B are symmetric traceless tensors, hence they carry the representation D^s of $SO(3)$; furthermore, the identities

$$\varepsilon_{b_1 j_1 k_1}\varepsilon_{b_2 j_2 k_2} F_{[j_1 k_1][j_2 k_2][\mu_3\nu_3]\ldots[\mu_h\nu_h]} = -F_{[0b_1][0b_2][\mu_3\nu_3]\ldots[\mu_h\nu_h]}$$

shows that two "magnetic" indices amount to two "electric" indices up to a sign, so that the $SO(3)$ tensors E and B contain all independent components of the higher Maxwell tensor.

Thus, a general field operator is of the form $F(f) = E(f^E) + B(f^B)$, where the test function is a pair

$$f(x) = (f^E_{b_1\ldots b_h}(x), f^B_{b_1\ldots b_h}(x))$$

of completely symmetric traceless tensors.

Also the higher Maxwell equations look the same as for $h = 1$, namely E and B are divergence-free and

$$\varepsilon_{abc}\nabla_a E_{bb_2\ldots b_h} = -\partial_t B_{cb_2\ldots b_h}, \quad \varepsilon_{abc}\nabla_a B_{bb_2\ldots b_h} = \partial_t E_{cb_2\ldots b_h} \qquad (4.14)$$

(which of course holds in every index).

Test functions that arise by smearing the Maxwell equations belong to the kernel of the two-point function, and hence are zero as elements of the one-particle Hilbert space. Thus, in the one-particle space, test functions of the form

$$(\nabla_b g_{\underline{b}}, 0) \doteq 0, \qquad\qquad (0, \nabla_b g_{\underline{b}}) \doteq 0,$$
$$(\varepsilon_{abc}\nabla_a g_{c\underline{b}}, 0) \doteq (0, -\partial_t g_{\underline{b}}), \qquad (0, \varepsilon_{abc}\nabla_a g_{c\underline{b}}) \doteq (\partial_t g_{\underline{b}}, 0). \qquad (4.15)$$

Because the higher Maxwell equations imply the wave equation, also

$$((\vec\nabla^2 - \partial_t^2)g^E, (\vec\nabla^2 - \partial_t^2)g^B) \doteq 0$$

are zero in the one-particle space.

4.2 Counting currents

The space of "test functions" for the fields restricted to the time axis is spanned by $f = (f^E_{\underline{b}}, f^B_{\underline{b}})$ where[3]

$$f^X_{\underline{b}}(x) = f^X_{\underline{b};\underline{a}}(t)\nabla_{\underline{a}}\delta(x) \equiv f^X_{b_1\ldots b_h;a_1\ldots a_k}(t) \cdot \nabla_{a_1}\ldots\nabla_{a_k}\delta(\vec x) \quad (X = E, B)$$

(summation over $\underline{a} = a_i\ldots a_k$ understood), $k = 0, 1, 2, \ldots$. We call T_k the subspace of such functions with a fixed number k of spatial derivatives and T the union of all the T_k.

---

[3]That Wightman fields can be restricted to $\vec x = 0$ is a result due to Borchers [3]. It ensures that the scalar product is well-defined on test functions involving $\delta(\vec x)$.

The space of the test functions $f_{\underline{b}}^X$, modulo the kernel of the scalar product, defines the one-particle Hilbert space $\mathcal{H}$ of the field strength F, and by Haag duality $K(O_I) = \overline{\{F(f)\Omega : f \in T,\ \mathrm{Supp}\, f \subset I\}} \subset H(O_I)$ cf. [3]. Furthermore, by conformal covariance, the modular group of the double cone subspace $H(O_I)$ implements a one-parameter group of conformal transformations fixing the time axis (see [16]) and any T_k (the whole representation $\tilde{U}|_{\mathrm{Möb}}$ fixes T_k, cf. (4.17)). Now one can see that $T + iT$ is cyclic in $\mathcal{H}$ since the scalar product in the Hilbert space $\mathcal{H}$ can be decomposed as $(f, g) = \int p_0 dp_0 \int_{p_0 \cdot S_2} (f, g)_p d\sigma$ where $p_0 \cdot S_2$ is the sphere of radius p_0, $d\sigma$ is the SO(3)-invariant measure on $p_0 \cdot S_2$ and $(\cdot, \cdot)_p$ is a quadratic form involving $2h$ factors of $p_\mu = (p_0, \vec{p}) = p_0(1, \vec{n}_\sigma)$.[4] Then, by Lemma 3.1 we have that $K(O_I) = H(O_I)$.

Because of the symmetry of the tensors E and B, it suffices to take $f_{\underline{b};\underline{a}}^X$ to be symmetric and traceless in the b-indices; and because of the wave equation, it suffices to take it also symmetric and traceless in the a-indices. Thus, the test functions carry (twice) the representation $D^s \otimes D^k$.

The one-particle Hilbert space is defined by taking the quotient by the null space, which is the kernel of the two-point function. Thus, we may identify test functions according to (4.15). In particular, every test function in T_k with coefficients $f_{\underline{b};\underline{a}}^X$ involving a factor $\delta_{b_i a_j}$ is zero in the one-particle space; and every test function in T_k with coefficients antisymmetric in a pair b_i, a_j is identified with (the time derivative of) a test function in T_{k-1}. Therefore, the one-particle Hilbert space for the restricted fields is spanned by the spaces $\widetilde{T}_k$ ($k = 0, 1, 2 \dots$) with elements

$$(f_{c_1 \dots c_{h+k}}^E, f_{c_1 \dots c_{h+k}}^B) \in \widetilde{T}_k$$

where $f_{\underline{c}}^X$ ($X = E, B$) are completely symmetric and traceless, hence carrying (twice) the representation D^{h+k} of $SO(3)$. All other subrepresentations of $D^h \otimes D^k$ belong to the null space. In particular $\widetilde{T}_k$ are mutually orthogonal.

We write the two-point function for $f = (f_{\underline{c}}^E, f_{\underline{c}}^B) \in \widetilde{T}_k$ as

$$(f, f)_k = \int \frac{p_0^2 \, dp_0}{p_0} \int d\sigma \left(\widehat{f}(p_0, \vec{p}), \widehat{f}(p_0, \vec{p}) \right)_p \tag{4.16}$$

where $\widehat{f}(p_0, \vec{p}) = (\widehat{f^E}_{c_1 \dots c_{h+k}}(p_0) p_{c_{h+1}} \cdots p_{c_{h+k}}, \widehat{f^B}_{c_1 \dots c_{h+k}}(p_0) p_{c_{h+1}} \cdots p_{c_{h+k}})$ are homogeneous polynomials of degree k in $\vec{p}$. Extracting powers of $|\vec{p}| = p^0$, this becomes

$$(f, f)_k = \int p_0^{1+2h+2k} \, dp_0 \int d\sigma \left(\widehat{f}(p_0, \vec{n}_\sigma), \widehat{f}(p_0, \vec{n}_\sigma) \right)_{(1, \vec{n}_\sigma)}.$$

The integration $d\sigma$ yields the scalar product for $D^{h+k} \oplus D^{h+k}$, while the Möbius transformations are characterized by the dependence on p^0.

The concluding argument is the same as in [7]: The time translations and dilations trivially restrict to the time axis by

$$P f(t) = i \partial_t f(t), \quad D f(t) = i(t \partial_t - (h + k)) f(t).$$

---

[4]By the Stone theorem polynomials are dense in the continuous functions on the sphere. Then (vector) continuous functions are dense in the L^2-space w.r.t. the scalar product $\int_{p_0 \cdot S_2} (f, g)_p d\sigma$.

The conformal inversion I acts geometrically on test functions by $(t, \vec{x}) \mapsto (-t, \vec{x})/(t^2 - \vec{x}^2)$, hence also its restricted action on the time axis is geometric by $t \mapsto -1/t$. Because it commutes with SO(3), it preserves the spaces $\widetilde{T}_k$ and must act on it as

$$(If)(t) = G(t)f(I(t)) \tag{4.17}$$

where $G(t)$ is a 2×2-matrix in the commutant of $D^{h+k} \oplus D^{h+k}$, possibly mixing the electric and magnetic components. Now the argument of the Lemma 3.6 applies, and we conclude that $G(t) = t^{2(h+k)}\mathbf{1}_2$, and the subgroup Möb $\times$ SO(3) of $\mathcal{C}$ acts on $\widetilde{T}_k$ as $U_{h+k+1} \otimes (D^{h+k} \oplus D^{h+k})$.

We have proved the following theorem.

Theorem 4.2. *Let U_h be the irreducible helicity-h representation of the Poincaré group. Let $U = U_h \oplus U_{-h}$, and $\widetilde{U}$ its extension to the conformal group $\mathcal{C}$, then*

$$\widetilde{U}|_{\textsf{Möb} \times \text{SO}(3)} = \bigoplus_{k=0}^{\infty} U^{(h+k+1)} \otimes (D^{h+k} \oplus D^{h+k}). \tag{4.18}$$

Corollary 4.3. *Let U_h be the irreducible helicity-h representation of the Poincaré group and $\widetilde{U}_h$ its extension to the conformal group $\mathcal{C}$, then*

$$\widetilde{U}_h|_{\textsf{Möb} \times \text{SO}(3)} = \bigoplus_{k=0}^{\infty} U^{(h+k+1)} \otimes D^{h+k}. \tag{4.19}$$

Proof. The PCT symmetry respects the Möb $\times$ SO(3) decomposition. Its anti-unitary implementation J intertwines U_h, U_{-h} and their restrictions to Möb $\times$ SO(3). Irreducible unitary sub-representations in $\tilde{U}|_{\textsf{Möb} \times \text{SO}(3)}$ of Möb $\times$ SO(3) are tensor products of the form $U^{j+1} \otimes \mathcal{D}^j$ that anti-unitarily extend to $\textsf{Möb}_2 \times \text{SO}(3)$. In particular, $\tilde{U}_h|_{\textsf{Möb} \times \text{SO}(3)}$ and $\tilde{U}_{-h}|_{\textsf{Möb} \times \text{SO}(3)}$ are unitarily equivalent, and by the decomposition in Theorem 4.2 we get the claim. $\qquad\square$

5 Trace class and split property for finite helicity fields

Definition 5.1. (Split Property) [11]. Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a *standard inclusion* of von Neumann algebras, i.e., Ω is a cyclic and separating vector for $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{N}' \cap \mathcal{M}$.

A standard inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is *split* if there exists a type I factor $\mathcal{B}$ such that $\mathcal{N} \subset \mathcal{B} \subset \mathcal{M}$.

A Poincaré covariant net $(\mathcal{A}, U, \Omega)$ satisfies the *split property* if the von Neumann algebra inclusion $(\mathcal{A}(O_1) \subset \mathcal{A}(O_2), \Omega)$ is split, for every compact inclusion of bounded causally closed regions $O_1 \Subset O_2$.

The following result relates the trace class property of the partition function in the first and second quantization nets.

Lemma 5.2. *[7, 22] Let $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator s.t. $0 \leq A < 1$, then $\text{Tr}\,\Gamma(A) < \infty$ iff $\text{Tr}\,A < \infty$, where Γ is the second quantization functor.*

The next proposition relates the trace class and the split properties of conformal nets on the circle.

Proposition 5.3. *[7] Let $\mathcal{A}$ be a von Neumann algebra net on the circle satisfying the trace class condition*

$$\text{Tr } e^{-\beta L_0} < \infty \quad \text{for every } \beta > 0,$$

then every inclusion $\mathcal{A}(I) \subset \mathcal{A}(\tilde{I})$, with $I \Subset \tilde{I}$, is a split inclusion.

The results of the previous section allow us to conclude

Proposition 5.4. *Let U_h be a finite helicity representation of $\mathcal{P}_+^\uparrow$ and $\widetilde{U}_h$ its extension to the conformal group $\mathcal{C}$. Consider the restriction $\widetilde{U}_h|_{\text{Möb}}$ and let L_0 be the conformal Hamiltonian, i.e., the generator of the rotation subgroup of Möb. Then $e^{-\beta L_0}$ is a trace class operator.*

Proof. By Corollary 4.3 any representation of highest weight $n+1$ in the decomposition of $U_h|_{\text{Möb} \times SO(3)}$ appears with multiplicity equal to the dimension of $\mathcal{D}^n$ when $n \geq s$:

$$\widetilde{U}_h|_{\text{Möb} \times SO(3)} \simeq \bigoplus_{k=0}^\infty U^{(h+k+1)} \otimes \mathcal{D}^{h+k}.$$

Furthermore, the trace of L_0 in U^{h+k+1} is equal to $\frac{e^{-(h+k+1)\beta}}{1-e^{-\beta}}$. We conclude that

$$\text{Tr } (e^{-\beta L_0}) = \sum_{n=h}^\infty (2n+1) \frac{e^{-(n+1)\beta}}{1 - e^{-\beta}},$$

which converges for all $\beta > 0$ as before. $\qquad\qquad\square$

Proposition 5.5. *Let $\mathcal{A}_h$ be the helicity-h free net of von Neumann algebras (whose one-particle space carries the representation $U_h \oplus U_{-h}$ if $h > 0$) and $I \mapsto \mathcal{A}_h(I) \doteq \mathcal{A}_h(O_I)$ be its restriction to the time axis. Then $\mathcal{A}_h(I) \subset \mathcal{A}_h(\tilde{I})$ is a split inclusion when $I \Subset \tilde{I}$.*

Proof. The net $\mathcal{A}_h$ is the second quantization of the BGL standard subspace net H_h associated to $U = U_h \oplus U_{-h}$. By Lemma 5.2 and Proposition 5.4, we have that $\text{Tr } \Gamma(e^{-\beta L_0}) < \infty$, thus the net

$$I \mapsto \mathcal{A}_h(O_I)$$

satisfies the split property, by Proposition 5.3. $\qquad\qquad\square$

Theorem 5.6. *The free finite helicity fields satisfy the split property.*

Proof. For inclusion of algebras related to double cones on the time axis, we conclude by Proposition 5.5.

For a general inclusion of double cones $O \Subset \tilde{O}$, choose a Poincaré transformation g such that $g(\tilde{O}) = O_{\tilde{I}}$ is a double cone on the time axis. Then there is an inclusion $O_I \Subset O_{\tilde{I}}$ of another double cone on the time axis such that $g(O) \subset O_I$. Then $\mathcal{A}_h(g(O)) \subset \mathcal{A}_h(O_{\tilde{I}}) = \mathcal{A}_h(g(\tilde{O}))$ is split because $\mathcal{A}_h(g(O)) \subset \mathcal{A}_h(O_I)$, and hence $\mathcal{A}_h(O) \subset \mathcal{A}_h(\tilde{O})$ is split by covariance. $\qquad\qquad\square$

As a corollary of Proposition 5.4 we also have the L^2-nuclearity property, which is stronger than the split property.

Corollary 5.7. *(L^2-nuclearity) Let $\mathcal{A}_h$ be the helicity-h free net of von Neumann algebras and $I \mapsto \mathcal{A}_h(I) \doteq \mathcal{A}_h(O_I)$ be its restriction to the time axis. Then for $I \Subset \tilde{I}$ the operator $\Delta_{\mathcal{A}(\tilde{I}),\Omega}^{1/4} \Delta_{\mathcal{A}(I),\Omega}^{-1/4}$ is trace class.*

The proof of the corollary is analogous to the one given in [7].

6 Outlook: torwards a new construction of finite helicity fields

Disjoint unitary representations of a given locally compact group G can have unitary equivalent restrictions to subgroups. This fact can be used to reconstruct inequivalent representations of G, by perturbing generators in the complement of a subgroup $H \subset G$. In [12] the authors proved that inequivalent highest weight representations of the Möb group have unitary equivalent restrictions to the translation-dilation subgroup, cf. Sect. 2.3. In particular one can recover the full Möb representation $U^{(h)}$ with lowest weight-h by perturbing the conformal inversion operator of the lowest weight-1 representation $U^{(1)}$. On the other hand, the covariance of associated nets is not preserved in this perturbation procedure. For instance, one can see that $U^{(h)}$ acts covariantly only a subnet of the $U(1)$-current (which anyway coincides with the $U(1)$-current on half-lines) [12].

In this paper we established the split property for free finite helicity fields. The fundamental step is the factor decomposition of the restriction of $\tilde{U}_h$, the extension of U_h to the conformal group $\mathcal{C}$, to the subgroup Möb $\times$ SO(3). The rotation group SO(3) is a type I group, hence irreducible representations of Möb$\times$SO(3) have to be tensor products $U^{(h)} \otimes \mathcal{D}^s$, where $U^{(h)}$ is the lowest weight-h representation of Möb and $\mathcal{D}^s$ is the spin-s representation of SO(3). By inspection of the decomposition of $\tilde{U}_h|_{\text{Möb}\times\text{SO}(3)}$ in Corollary 4.3, we observe that $\tilde{U}_{h_1}|_{\text{Möb}\times\text{SO}(3)}$ is a sub-representation of $\tilde{U}_{h_2}|_{\text{Möb}\times\text{SO}(3)}$ when $h_1 - h_2 \in \mathbb{Z}$ and $h_1 \geq h_2$.

One can think of a perturbation argument. Consider the projection P_h on the subspace supporting $\bigoplus_{k=0}^{h-1}(U^{k+1} \otimes \mathcal{D}^k)$ and cut $\tilde{U}_0|_{\text{Möb}\times\text{SO}(3)}$ along the complementary space $1 - P_h$. By Corollary 4.3 the representation $U_0|_{\text{Möb}\times\text{SO}(3)}(1 - P_h)$ extends to a representation of helicity h by redefining the spatial translations, suitably perturbing the translation generators in the scalar representation on $(1 - P_h)\mathcal{H}_0$. Namely, the spatial translations together with the time translations and the conformal inversion, contained in Möb, generate the conformal group. This can be further seen by looking at the proof of Proposition A.4, where we disintegrate the spectrum in rotation-translation invariant fibers, and (A.21) shows that $W_{h,p_0}|_{\text{SO}(3)} \leq W_{k,p_0}|_{\text{SO}(3)}$, for $k \leq h$. Thus one can address the perturbation argument already at the level of the euclidean subgroup, cf. Appendix A.

Let us comment on inclusions of standard subspace nets on the time axis. Firstly, the BGL-net associated to the $\mathcal{P}_+^\uparrow$-representation U_0 extends to a conformal net, and the Bisognano-Wichmann property for boosts and dilations is a consequence of conformal covariance, cf. [10]. Then, we note that the projection $P_{h+1} - P_h$ commutes with $U_0|_{\text{Möb}}$ (and with $U_0|_{\text{Möb}\times\text{SO}(3)}$). In particular, the net on the time axis $I \mapsto H_0(O_I)$ decomposes as the

direct sum of Möb-covariant nets of subspaces

$$I \mapsto H_0(O_I) = \bigoplus_{h=0}^{\infty} (P_{h+1} - P_h) \, H_0(O_I)$$

according to (4.19). Indeed, by the Bisognano-Wichmann property the modular groups of the interval subspaces implement interval dilations, the interval modular conjugations implement the PCT symmetry, and it is easy to see that the Tomita operators of the interval subspaces commute with $P_{h+1} - P_h$.

Once we identify the representations

$$(\tilde{U}_h \oplus \tilde{U}_{-h})|_{\mathsf{Möb} \times \mathrm{SO}(3)} = ((1 - P_h) \oplus (1 - P_h)) \, (\tilde{U}_0 \oplus \tilde{U}_0)|_{\mathsf{Möb} \times \mathrm{SO}(3)} \, ((1 - P_h) \oplus (1 - P_h)) \,,$$

by Proposition 3.5, we can also identify $H_h(I)$ as a subnet of $H_0(I) \oplus H_0(I)$: take two copies of the massless scalar one particle net $(U_0 \oplus U_0, H_0 \oplus H_0)$ and consider the net on the time axis $I \mapsto H_0(O_I) \oplus H_0(O_I)$; then consider the $\mathsf{Möb} \times \mathrm{SO}(3)$ invariant projections $(1 - P_h) \oplus (1 - P_h)$ and the new net on the time axis

$$I \mapsto H_h(I) \doteq ((1 - P_h) \oplus (1 - P_h)) \, (H_0(O_I) \oplus H_0(O_I)) \,,$$

undergoing the $\mathsf{Möb}$ (and $\mathsf{Möb} \times \mathrm{SO}(3)$)-action through $(\tilde{U}_h \oplus \tilde{U}_{-h})|_{\mathsf{Möb} \times \mathrm{SO}(3)}$. The projection $1 - P_h$ does not commute with the U_0-translations since U_0 is irreducible, and on the subspace $((1 - P_h) \oplus (1 - P_h)) \, (\mathcal{H}_0 \oplus \mathcal{H}_0)$ one has to define new translations to obtain the $U_h \oplus U_{-h}$ representation of the Poincaré group (the group generated by $\mathsf{Möb} \times SO(3)$ and space translations contains the Poincaré group). Afterwards, one can define by covariance double cone subspaces and the helicity-h free net of standard subspaces, namely

$$H_h(O) \doteq (U_h(g) \oplus U_{-h}(g)) \, H_h(O_I)$$

for a general double cone $O = gO_I$. It remains an interesting open problem to explicitly provide or characterize the necessary perturbation of the U_0-translations in order to obtain the U_h-translations on the proper subspace.

This further suggests another way of constructing finite helicity free nets. One can start with the representation of $\mathsf{Möb} \times \mathrm{SO}(3)$ in the right-hand side of (4.19). It extends to the representation of the Poincaré group of helicity h or $-h$. Consider two copies of such a $\mathsf{Möb} \times \mathrm{SO}(3)$-representation, and the associated one-particle net on the line can be identified with the time axis theory of the helicity-h free net. Then there is a proper choice of the translation generators and the PCT operator which allows to construct the free net on the full Minkowski space by covariance.

A   Appendix: Restriction of finite helicity representations to the euclidean subgroup

We comment on the restriction of finite helicity representations to the euclidean group.

Definition A.1. [25] Let G be a separable locally compact group.

Closed subgroups G_1 and G_2 of G are said to be *regularly related* if there exists a sequence $E_0, E_1, E_2, \ldots$ of measurable subsets of G each of which is a union of $G_1 : G_2$ double cosets

such that E_0 has Haar measure zero and each double coset not in E_0 is the intersection of the E_j which contain it.

Because of the correspondence between orbits of G/G_1 under G_2 and double $G_1 : G_2$ cosets, G_1 and G_2 are regularly related if and only if the orbits outside of a certain set of measure zero form the equivalence classes of a measurable equivalence relation.

Consider the map $s : G \to G_1 \backslash G / G_2$ carrying each element of G into its double coset. Then equip $G_1 \backslash G / G_2$ with the quotient topology given by s and consider a finite measure μ on G which is in the same measure class of the Haar measure. It is possible to define $\overline{\mu}$ on the Borel sets of $G_1 \backslash G / G_2$ by $\overline{\mu}(E) = \mu(s^{-1}(E))$. We shall call $\overline{\mu}$ an *admissible measure* in $G_1 \backslash G / G_2$. The definition is well posed since any two of such measures have the same null measure sets.

We recall two well-known theorems.

Theorem A.2 (Mackey's subgroup Theorem). *[26]. Let G_1, G_2 regularly related in G. Let $\pi \in \mathrm{Rep}(G_1)$. For each $x \in G$ consider $G_x = G_2 \cap (x^{-1}G_1 x)$ and set*

$$V_x = \mathrm{Ind}_{G_x \uparrow G_2}(\pi \circ \mathrm{ad}\, x).$$

Then V_x is determined to within equivalence by the double coset $\overline{x}$ to which x belongs. If ν is an admissible measure on $G_1 \backslash G / G_2$, then

$$\left(\mathrm{Ind}_{G_1 \uparrow G}\pi\right)|_{G_2} \simeq \int_{G_1 \backslash G / G_2} V_{\overline{x}}\, d\nu(\overline{x}).$$

If G is a compact group, let π and ρ be two unitary representations of G, we shall denote with $\mathcal{C}(\pi, \rho)$ the space of intertwining operators of the representations π and ρ and with $\mathrm{mult}(\pi, \rho)$ the multiplicity (of the unitary class) of π in ρ.

Theorem A.3 (Frobenius Reciprocity theorem). *[13]. Let G a compact group, H a closed subgroup, π a unitary representation of the group G, and σ an irreducible unitary representation of H. Then,*

$$\mathcal{C}(\pi, \mathrm{Ind}_{H \uparrow G}(\sigma)) \simeq \mathcal{C}(\pi|_H, \sigma) \quad and \quad \mathrm{mult}\,(\pi, \mathrm{Ind}_{H \uparrow G}(\sigma)) = \mathrm{mult}\,(\pi|_H, \sigma).$$

In this section we shall indicate with χ a one dimensional representation (a character) of an abelian group. Let $E(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$ be the inhomogeneous symmetry group of n-dimensional Euclidean space. The universal covering is the semi-direct product $\widetilde{E}(n) = \widetilde{\mathrm{SO}}(n) \ltimes \mathbb{R}^n$. Representations are obtained by induction. Consider a character χ_q in the dual of the translation group and its orbit σ_q through the dual action of $\widetilde{E}(n)$. We shall call Stab_q and $\overline{\mathrm{Stab}}_q = \mathrm{Stab}_q \ltimes \mathbb{R}^n$ the stabilizers of χ_q through the $\widetilde{\mathrm{SO}}(n)$ and $\widetilde{E}(n)$ actions, respectively. Note that the dual action of the translations is trivial on χ_q. When there is no ambiguity we will write q instead of χ_q.

There are two main families of irreducible representations (cf. e.g. [29, 13]):

- $U = \mathrm{Ind}_{\overline{\mathrm{Stab}}_0 \uparrow \widetilde{E}(n)}\chi_0 V = V$ is induced from a product of the trivial character χ_0 of $\mathbb{R}^n$ and an irreducible representation V of $\mathrm{Stab}_0 = \widetilde{\mathrm{SO}}(n)$. Thus U is the irreducible representation V of $\widetilde{\mathrm{SO}}(n)$ lifted to $\widetilde{E}(n)$, trivial on translations;

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- $U = \mathrm{Ind}_{\overline{\mathrm{Stab}_q} \uparrow \widetilde{E}(n)} \chi_q V'$ is induced from a product of a nontrivial character χ_q of $\mathbb{R}^n$ and an irreducible representation V' of Stab_q. In such a case orbits are spheres of radius $r = |q|$ and up to unitary equivalence it is possible to choose $q = (\mathbf{0}, r)$ where $\mathbf{0}$ is the null vector in $\mathbb{R}^{n-1}$.

In the three-dimensional euclidean case, if $q = (0, 0, r)$ with $r > 0$ then $\mathrm{Stab}_q = U(1)$, double covering of $\mathrm{SO}(2)$. Induced representations are of the form

$$W_{h,r} = \mathrm{Ind}_{\overline{\mathrm{Stab}_q} \uparrow \widetilde{E}(3)} \chi_q \chi_h, \qquad h \in \frac{1}{2}\mathbb{Z},$$

where χ_h is the $2h$-character $U(1)$-representation and q defines a character of $\mathbb{R}^3$ of length r. The induced representation acts on the Hilbert space $L^2(S_r, dp\,\delta(\mathbf{p}^2 - r^2))$ where S_r is the sphere with center in the origin and radius r.

Proposition A.4. *Let U_h be a massless helicity-h representation. Consider the restriction of U_h to $T \times E(3)$, where T is the time-translation group, then*

$$U_h|_{T \times E(3)} = \int_{\mathbb{R}^+} dp_0 \left(\chi_{p_0} \otimes W_{h,p_0} \right). \tag{A.20}$$

Furthermore,

$$W_{h,p_0}|_{\mathrm{SO}(3)} = \bigoplus_{l=|h|}^{\infty} \mathcal{D}^l \tag{A.21}$$

and

$$U_h|_{T \times \mathrm{SO}(3)} = \bigoplus_{l=|h|}^{\infty} \int_{\mathbb{R}^+} dp_0 \left(\chi_{p_0} \otimes \mathcal{D}^l \right) \tag{A.22}$$

Proof. We prove the proposition in the bosonic case, namely $h \in \mathbb{Z}$. The proof is analogous in the Fermionic case.

Let $q = (1, 0, 0, 1)$, with the definitions in Sect. 2.2, the helicity-h representation is

$$U_h = \mathrm{Ind}_{\overline{\mathrm{Stab}_q} \uparrow \mathcal{P}_+^\uparrow} \chi_q V_h$$

where $\overline{\mathrm{Stab}_q} = E(2) \ltimes \mathbb{R}^4 \subset \mathcal{P}_+^\uparrow$. Let T be the time translation group. When we restrict U_h to $T \times E(3)$, we get

$$U_h|_{T \times E(3)} = \int_{\mathbb{R}^+} dp_0 \, \chi_{p_0} \, U_{p_0} \tag{A.23}$$

where U_{p_0} are representations of $E(3)$ of radius p_0. This follows since

$$(U_h(a, A)\phi)(p) = e^{ia \cdot p} V_h(B_p^{-1} A B_{A^{-1}p}) \phi(A^{-1}p), \quad (a, A) \in \mathcal{P}_+^\uparrow, \phi \in L^2(\partial V_+, \theta(p_0)\delta(p^2)d^4p),$$

and we can choose $B_p^{-1} = \Lambda_3(-\ln p_0)R_p$, where $p \mapsto R_p$ is a Borel map from the $\mathbb{R}^3$-sphere S_{p_0} of radius p_0 (we are considering the set $(p_0, S_{p_0}) \subset \mathbb{R}^{1+3}$) to $\mathrm{SO}(3)$ such that, for any p, $R_p\, p = q_{p_0} \doteq (p_0, 0, 0, p_0)$ and Λ_3 is the x_0-x_3 boost s.t. $\Lambda_3(-\ln p_0)q_{p_0} = q = (1, 0, 0, 1)$ (cf. [29]). Thus, with $U_{p_0} = e^{ia \cdot p} V_h(R_p^{-1} A R_{A^{-1}p})\psi(A^{-1}p)$ where $(a, A) \in E(3)$ and $\psi \in L^2((p_0, S_{p_0}), \delta(p^2)\delta(p_0)d^4p)$, the direct integral representation of $T \times E(3)$ in the right hand side of (A.23) extends to the representation of the Poincaré group U_h.

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Now, let $\delta_t : p \mapsto e^\lambda p$ be the dilation group, by dilation covariance of U_h

$$U|_{T \times E(3)} = \int_{\mathbb{R}^+} dp_0 \, U_{p_0} \chi_{p_0} = U(\delta_t) \left(\int_{\mathbb{R}^+} dp_0 \, U_{p_0} \chi_{p_0} \right) U(\delta_t)^* = \int_{\mathbb{R}^+} dp_0 \, U_{\delta_{-t}p_0} \chi_{\delta_{-t}p_0},$$

thus $U_{p_0} \chi_{p_0} = U_{\delta_{-t}p_0} \chi_{\delta_{-t}p_0}$ for a.e. $\lambda \in \mathbb{R}$. In particular, U_{p_0} is irreducible for almost every $p_0 \in \mathbb{R}^+$ because the U_h-translation algebra is multiplicity free.

Since the stabilizer of q_{p_0} under the $(T \times E(3))$-action is contained in $E(2) \ltimes \mathbb{R}^4 \subset \mathcal{P}_+^\uparrow$ (the stabilizer of q_{p_0} under the Poincaré action) and V_h is trivial on $E(2)$-translations, then one can see that for almost every $p_0 \in \mathbb{R}^+$, $U_{p_0} = \mathrm{Ind}_{\mathrm{SO}(2) \ltimes \mathbb{R}^3 \uparrow E(3)} \chi_h \chi_{q_{p_0}}$ and we get

$$U_h|_{E(3)} = \int_{\mathbb{R}^+} dp_0 \, W_{h,p_0} \chi_{p_0}.$$

Now, we apply Theorem A.2 to W_{h,p_0} with $G = E(3)$, $G_1 = \mathrm{SO}(2) \ltimes \mathbb{R}^3$, $G_2 = \mathrm{SO}(3)$ (note that $G_1 \backslash G / G_2 = \mathbf{1}$). By Theorem A.3, we get the second statement, i.e.

$$W_{h,p_0}|_{\mathrm{SO}(3)} = \bigoplus_{l=|h|}^{\infty} \mathcal{D}^l.$$

(A.21) does not depend on the radius p_0, thus we conclude (A.22). $\square$

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