NON-TRIVIAL BUNDLES AND ALGEBRAIC CLASSICAL FIELD THEORY

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Abstract. Inspired by the recent algebraic approach to classical field theory, we propose a more general setting based on the manifold of smooth sections of a non-trivial fiber bundle. Central is the notion of observables over such sections, *i.e.* appropriate smooth functions on them. The kinematic will be further specified by means of the Peierls brackets, which in turn are defined via the causal propagators of linearized field equations. We shall compare the formalism we use with the more traditional ones.

Contents

1. Introduction

The aim of the present paper is to generalize the recent treatment of relativistic classical field theory [BFR19], seen as a Lagrangian theory based on (non linear) functionals over the infinite dimensional configuration space, to the case where the latter is made of sections of general bundles.

There are few mathematical treatments of classical field theories, all finding inspirations and drawing ideas from two main sources, the Hamiltonian and Lagrangian formalisms of classical mechanics. If one intends also to include relativistic phenomena then there remain essentially only two rigorous frameworks, both emphasizing the geometric viewpoint: the multisymplectic approach (see [Got+98], [GIM04], [FR05]) and another related to the formal theory of partial differential equations (see [FF03], [Kru15]). They have several points in common and there is now a highly developed formalism leading to rigorous calculus of variations.

Physicists look at Lagrangian classical field theories with more interest. This clearly amounts to develop a formalism in which one treats the intrinsic infinite dimensional degrees of freedom of the configuration spaces. Both the previously cited frameworks use ingenuous ideas to avoid the direct treatment of infinite dimensional situations. On the other hand, there exists another treatment of classical mechanics that emphasises more the algebraic and the analytic structures and is intrinsically infinite dimensional, this is named after the pioneering works of von Neumann ([Neu32]) and Koopman ([Koo31]) and works directly in Hilbert spaces. If one is willing to generalise this last setting to field theories finds immediately an insurmountable difficulty, namely, a result by Eells and Elworthy (see [EE68], [EE70]) constrains a configuration space, viewed as a second countable Hilbert manifold, to be smoothly embedded into its ambient space, i.e., it is just an open subset of the Hilbert space. Hence, we need to bypass this fact of life and find a clever replacement.

This task was done in the last decade, in which another treatment was developed that drew inspirations from perturbative quantum field theories in the algebraic fashion [BDF07] and which is closer in spirit to the von Neumann-Koopman formalism. Indeed, it emphasises more the observables's point of view and deals directly with the configuration space as an infinite dimensional manifold but modelled now over locally convex spaces. Here one relies heavily on the clarifications given in the last thirty years about the most appropriate calculus on such complicated spaces (see, *e.g.*, [Bas64; KM97]). Based on such ideas, and moreover using as inputs also some crucial notions belonging to microlocal analysis, one of the authors (RB) in collaboration with Klaus Fredenhagen and Pedro Lauridsen Ribeiro [BFR19], have formalised the new treatment for the case of scalar fields on globally hyperbolic spacetimes. As advertised above, it is one of the main aims of the present paper to generalise such treatment to the more complicated situation in which fields are sections of fibre bundles. At first sight the idea looks straightforward to implement, however it contains some not trivial subtleties whose treatment needs a certain degree of care. Indeed, in our general setting, images of the fields are never linear spaces and moreover, the global configuration space has only a manifold structure. This forces us to generalize many notions like the support of functionals, or their central notion of locality/additivity, over configuration space, which can be given in two different formulations, one global that uses the notion of relative support already used in [BFR16] and a local one that uses the notion of charts over configuration space seen as a infinite dimensional manifold. It is gratifying that both notions give equivalent results, as shown *e.g.* in Proposition 3.8.

We summarize the content of this article.

Section 2 is devoted to the geometrical tools used in the rest of the article: we introduce the classical geometrical formalism based on jet manifolds and then the infinite dimensional formalism.

Section 3 focuses on the definition of observables, their support, and the introduction to various classes of observables depending on their regularity. In particular two of this classes admit a Γ^∞-*local* characterization, i.e. local in the sense of the manifold structure of the space of sections. In the end we introduce the notion of *generalized Lagrangian*, essentially showing that each Lagrangian in the standard geometric approach is a Lagrangian in the algebraic approach as well. We then discuss how linearized field equations are derived from generalized Lagrangians.

Section 4 begins with some preliminaries about normal hyperbolicity and normally hyperbolic operators. Here generalized Lagrangians of second order which are normally hyperbolic, as it is in the case of wave maps, play a major role. We then show the existence of the causal propagator which in turn is used in Definition 4.5 to define the Poisson bracket on the class of *microlocal functionals*. Then we enlarge the domain of the bracket to the so called *microcausal functionals*, defined by requiring a specific form of the wave front set of their derivatives. Finally Proposition 4.6, Theorems 4.10, 4.11 and 4.12 establish the Poisson $*$-algebra of microcausal functionals.

Section 5 presents of results that culminate in Theorem 5.3, which establishes that microcausal functionals can be given the topology of a nuclear locally convex space. Furthermore Propositions 5.4 and 5.5 give additional properties concerning this space and its topology. We conclude the section by defining the on-shell ideal with respect to the Lagrangian generating the Peierls bracket and the associated Poisson $*$-algebraic ideal.

Eventually in section 6 we briefly show how to adapt the previous results to the case of wave maps.

2. Geometrical setting

2.1. **Field theoretical preliminaries.** Let M be a smooth m-dimensional manifold, suppose that there is a smooth section g of $T^*M \vee T^*M \to M)$, where $\vee$ denotes the symmetric tensor product, such that its signature is $(-,+,\ldots,+)$; then (M,g) is called a Lorentz manifold and g its Lorentzian metric. If we take local coordinates $(U,\{x^\mu\})$ and denote $d\sigma(x) \doteq dX^1 \wedge \ldots \wedge dx^m$, then $d\mu_g(x) = \sqrt{|g(x)|}d\sigma(x)$ is the canonical volume element of M, where as usual we denote by $g(x)$ the determinant of g calculated at $x \in M$. Any Lorentzian metric g induces the so called musical isomorphisms $g^\sharp : TM \to T^*M : (x,v) \mapsto (x, g_{\mu\nu}v^\mu dx^\nu)$, $g^\sharp : T^*M \to TM : (x,\alpha) \mapsto (x, g^{\mu\nu}\alpha_\mu\partial_\nu)$, where $\{dx^\mu\}_{\mu=1,\ldots,m}$ is the standard basis of T^*_xM in local coordinates $(U,\{x^\mu\})$, and $\{\partial_\mu\}_{\mu=1,\ldots,m}$ the dual basis of T_xM.

Given any Lorentzian manifold (M,g), a non zero tangent vector $v_x \in T_xM$ is *timelike* if $g_x(v_x,v_x) < 0$, *spacelike* if $g_x(v_x,v_x) > 0$, *lightlike* if $g_x(v_x,v_x) = 0$; similarly a curve $\gamma : \mathbb{R} \to M : t \mapsto \gamma(t)$ is called *timelike* (resp. *lightlike*, resp. *spacelike*) if at each $t \in \mathbb{R}$ its tangent vector is *timelike* (resp. *lightlike*, resp. *spacelike*), a curve that is either timelike or lightlike is called *causal*. We denote the cone of timelike vectors tangent to $x \in M$ by $V_g(x)$. A Lorentzian manifold admits a *time orientation* if there is a global timelike vector field T, then timelike vectors $v \in T_xM$ that are in the same connected component of $T(x)$ inside the light cone, are called *future directed*. When an orientation is present we can consistently split, for each $x \in M$, the set $V_g(x)$ into two disconnected components $V_x^+(x) \cup V_g^-(x)$ calling them respectively the sets of *future directed* and *past directed* tangent vectors at x. Given $x,y \in M$, we say that $x \ll y$ if there is a future directed timelike curve joining x to y, and $x \leq y$ if there is a causal curve joining x to y. We denote $I^+_M(x) = \{y \in M : x \ll y\}$, $I^-_M(x) = \{y \in M : x \gg y\}$, $J^+_M(x) = \{y \in M : x \leq y\}$, $J^-_M(x) = \{y \in M : x \geq y\}$ and call them respectively the *chronological future*, *chronological past*, *causal future*, *causal past* of x. (M,g) is said to be *globally hyperbolic* if M is causal, i.e. there are not closed causal curves on M and the sets $J_M(x,y) \doteq J^+_M(x) \cap J^-_M(y)$ are compact for all $x,y \in M$. Equivalently, M is globally hyperbolic if there is a smooth map $\tau : M \to \mathbb{R}$ called *temporal function* such that its level sets, Σ_t are Cauchy hypersrfaces, that is every inextensible causal curve intersects

Σ_t exactly once. A notable consequence is that any globally hyperbolic manifold M has the form $\Sigma \times \mathbb{R}$ for some, hence any, Cauchy hypersurface Σ. We point to [ONe83] for details.

A *fiber bundle* is a quadruple (B, π, M, F), where B, M, F are smooth manifold called respectively the *bundle*, the *base* and the *typical fiber*, such that: (i) $\pi : B \to M$ is a smooth surjective submersion; (ii) there exists an open covering of the base manifold M, $\{U_\alpha\}_{\alpha\in A}$ admitting, for each $\alpha \in A$, diffeomorphisms $t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$, called trivializations, having $pr_1 \circ t_\alpha\left(\pi^{-1}(U_\alpha)\right) = U_\alpha$, and $t_\alpha(\pi^{-1}(x)) \simeq F$ for all $x \in M$.

We denote by

$$\Gamma^\infty(M \leftarrow B) = \{\varphi : M \to B, \text{ smooth } : \pi \circ \varphi = id_M\} \tag{1}$$

the space of *sections* of the bundle. In physical terminology the latter are also called *field configurations* or simply *fields*. We shall later on put a topology and then an infinite dimensional manifold structure on $\Gamma^\infty(M \leftarrow B)$. We stress that for a generic bundle the space of global smooth sections might be empty (*e.g.* for nontrivial principal bundles), therefore we assume that our bundles do posses them. Indeed that is the case whenever we are considering trivial bundles, vector bundles, moreover, the bundle of connections on a manifold M is a natural bundle whose global sections are ordinary connections (see Section 14 in [KMS13] for the definition of natural bundles).

Using trivialization it is possible to construct charts of B via those of M and F. We call those *fibered coordinates* and denote them by (x^μ, y^i) with the understanding that Greek indices denotes the base coordinates and Latin indices the standard fiber coordinates. Given two fiber bundles (B_i, π_i, M_i, F_i), $i = 1, 2$, we define a *fibered morphism* as a pair (Φ, ϕ), where $\Phi : B_1 \to B_2$, $\phi : M_1 \to M_2$ are smooth mappings, such that $\pi_2 \circ \Phi = \phi \circ \pi_1$, this is why sometimes ϕ is called the base projection of Φ.

Vector bundles are particular fiber bundles whose standard fibers are vector spaces and their transition mappings act on fibers as transformations of the general Lie group associated to the standard fibers. We denote coordinates of these bundles by $\{x^\mu, v^i\}$. Suppose that (E, π, M, V), (F, ρ, M, W) be vector bundles over the same base manifold, then it is possible to construct a third vector bundle $(E \otimes F, \pi \otimes \rho, M, V \otimes W)$ called the tensor product bundle whose standard fiber is the tensor product of the standard fibers of the starting bundles.

In the sequel we will use particular vector fields of B, they are called *vertical vector fields* and are defined by $\mathfrak{X}_{\text{vert}}(B) = \{X \in \Gamma^\infty(B \leftarrow TB) : T\pi(X) = 0\}$, where $\pi : B \to M$ is the bundle projection and T denotes the tangent functor. We will denote by $\Phi_t^X : B \to B$ the flow of any vector field on B, and assume in the rest of this work that the parameter t varies in an appropriate interval which has been maximally extended. Note that if $X \in \mathfrak{X}_{\text{vert}}(B)$, then Φ_t^X is a fibered morphism whose base projection is the identity over M. Vertical vector fields can be seen as sections of the *vertical vector bundle*, $(VB \doteq \ker(T\pi), \tau_V, B)$ which is easily seen to carry a vector bundle structure over B. Another construction that we shall use repeatedly is that of *pullback bundles*, *i.e.* given a fiber bundle B as before and a smooth map $\psi : M \to N$, we can describe another bundle over N with the same typical fibers as the original fiber bundle and with total space defined by $\psi^*B \doteq \{(n, b) \in N \times B |\ \psi(n) = \pi(b)\}$ and projection $\psi^*\pi \doteq \mathrm{pr}_1 \restriction_{\psi^*B}$. It is easy to show local trivializations by which ψ^*B can be seen as a smooth submanifold of $N \times B$. One says that this is the *pullback bundle of B along ψ*.

An important notion necessary to the geometric framework of classical field theories are jet bundles. Heuristically they geometrically formalize PDEs. For general references see [KMS13] chapter IV section 12 or [Sau89]. Rather then giving the most general definition, we simply recall the bundle case. Given any fiber bundle (B, π, M, F), two sections, φ_1, φ_2 are kth-order equivalent in $x \in M$, which we write, $\varphi_1 \sim_x^k \varphi_2$ if for all $f \in C^\infty(B)$, $\gamma \in C^\infty(\mathbb{R}, M)$ having $\gamma(0) = x$, the Taylor expansions at 0 of order k of $f \circ \varphi_1 \circ \gamma$ and $f \circ \varphi_2 \circ \gamma$ coincide. The relation $\sim_x^k$ becomes an equivalence relation and we denote by $j_x^k\varphi$ the equivalence class with respect to φ. Letting $J_x^kB \doteq \Gamma_x^\infty(M \leftarrow B)/\sim_x^k$, where $\Gamma_x^\infty(M \leftarrow B)$ are the germs of local sections of B defined on a neighborhood of x, the kth order *jet bundle* is then

$$J^kB \doteq \bigsqcup_{x\in M} J_x^kB.$$

The latter inherits the structure of a fiber bundle with base either M, B or any J^lB with $l < k$. If $\{x^\mu, y^j\}$ are fibered coordinates on B, then we induce fibered coordinates $\{x^\mu, y^j, j_\mu^j, \ldots, y^j_{\mu_1\ldots\mu_k}\}$ where Greek indices are symmetric. The latter coordinates embody the geometric notion of PDEs. The family $\{(J^rB, \pi^r)\}_{r\in\mathbb{N}}$ with $\pi^r : J^rB \to M$ allows an inverse limit $(J^\infty B, \pi^\infty, \mathbb{R}^\infty)$ called the *infinite jet bundle* over M, it can be seen a fiber bundle whose standard fiber $\mathbb{R}^\infty$ is a Fréchet topological vector space. Its sections denoted by $j^\infty\varphi$ are called infinite jet prolongations.

2.2. **Topology and geometry of field configurations.** In this section we shall introduce some notation and recall some notions from infinite dimensional geometry that will be used throughout the paper.

Let M, N be Hausdorff topological spaces and let $C(M,N)$ the space of continuous mappings between the two spaces, the *compact open topology* or CO-topology has a subbasis given by sets of the form

$$\{f \in C(M,N) : f(K) \subseteq U\}$$

with $K \subset M$ compact and $U \subset N$ open. Since N is Hausdorff then the CO-topology is Hausdorff as well. Call $WO-$topology the one generated by a subbasis of the form $\{f \in C(M,N) : f(M) \subseteq U\}$ with $U \subset N$ open. The *graph topology* or WO^0-topology on $C(M,N)$ is the one generated by the subbasis

$$\{f \in C(M,N) : \mathrm{graph}(f) \subseteq U\}$$

with $U \in M \times N$ open. Equivalently we can require the mapping

$$\mathrm{graph} : C(M,N) \to (C(M, M \times N), WO - \text{topology})$$

to be a topological embedding. This topology is finer than the $CO-$topology which is therefore Hausdorff. In general the two topologies differ when M is not compact. Finally the *Whitney C^k topology*, or WO^k-topology is given by requiring

$$j^k : C^\infty(M,N) \to \big(C(M, J^k(M,N)), WO - \text{topology}\big)$$

to be an embedding. When $k = \infty$ we call the latter *Whytney topology* or WO^∞-topology. If M is second countable and finite dimensional, N is metrizable, by 41.11 in [KM97], given an exhaustion of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ in M and a sequence $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$, then a basis of open subset is given by

$$\big\{f \in C^\infty(M,N) : j^{k_n} f(M\backslash K_n) \subset U_n\big\}$$

where each $U_n \subset J^{k_n}(M,N)$ is an open subset.

When N is metrizable and M paracompact and second countable, if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $C^\infty(M,N)$ we can characterize its convergence to an $f \in C^\infty(M,N)$ in the following way:

- (i) $f_n \to f$ in the $WO^\infty-$topology ,
- (ii) $\forall n' \in \mathbb{N}\ \exists K_{n'} \subset M$ compact such that if $n \geq n'$ then $f_n|_{M\backslash K_{n'}} = f|_{M\backslash K_{n'}}$ and $f_n|_{K_{n'}} \to f|_{K_{n'}}$ uniformly with all its derivatives.

Suppose that M, N are smooth finite dimensional manifolds, set

$$f \sim g \Leftrightarrow \mathrm{supp}_f(g) = \overline{\{x \in M : f(x) \neq g(x)\}} \subset M \text{ is compact}\ .$$

We shall sometimes use the $\mathfrak{D}_F$-topology, which is the coarsest topology on $C^\infty(M,N)$ that is finer than the WO^∞-topology and for which the sets $U_f = \{g \in C^\infty(M,N) : g \sim f\}$ are open.

We briefly recall the notion of differential calculus on locally convex spaces (LCS) by Bastiani, see [Bas64; Sch22] for details.

Definition 2.1. Assume that E, F are LCS, let $U \subset E$, an open subset and $f : U \subset E \to F$ a map. We say that f is B-differentiable of order k on U, and use the symbol $f \in C^k_B(U)$, if for all $j \leq k$

- (i) the Gateaux derivatives

$$d^{(j)} f[x](y_1, \ldots, y_j) \doteq \frac{d^j}{dt_1 \ldots dt_j} f(x + t_1 y_1 + \ldots + t_k y_j)$$

  exist for each $x \in U$, $y_1, \ldots, y_j \in E$;
- (ii) the induced mappings

$$d^{(j)} f : U \times E \times \ldots \times E \to F$$

  are continuous.

We call B-smooth those mappings which are differentiable at all orders, symbolically they are generically named to be in C^∞_B.

By construction we see that the derivative is symmetric in the last entries. This differentiability behaves as expected in compositions, i.e. if A, B, C are LCS, $U \subset A$, $V \subset B$ open sets and $f : U \to B$, $g : V \to C$ be C^1_B maps such that $f(U) \subset V$, then

$$d^{(1)}(g \circ f)[x](y) = d^{(1)} g[f(x)]\left(d^{(1)} f[x](y)\right).$$

An important result that will used in the sequel is the fundamental theorem of calculus: if $f : U \subset E \to F$ is B-differentiable and U is convex, then

$$f(x_1 + x_2) - f(x_1) = \int_0^1 d^{(1)} f[x_1 + t x_2](x_2) dt \tag{2}$$

for all x_1, $x_2 \in U$.

Definition 2.2. Let X be a Hausdorff topological space, we say that X is a C^∞_B-manifold if

(i) there is a family (called atlas) $\{(U_\alpha, u_\alpha, E_\alpha)\}$, where $\{U_\alpha\}$ is an open cover of X, E_α are complete LCS and $u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subset E_\alpha$ is a homeomorphism for all $\alpha \in A$,

(ii) if $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ then
$$u_{\alpha\beta} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$$
is C^∞_B.

We call $(U_\alpha, u_\alpha, E_\alpha)$ a chart.

We shall now describe the infinite dimensional differential structure on the space of sections $\Gamma^\infty(M \leftarrow B)$ of a fiber bundle (B, π, M, F). The first step is to find a suitable topology for the space of mappings. We note that $\Gamma^\infty(M \leftarrow B) \subset C^\infty(M, B)$, since it is the subset of mappings which are left inverses to the bundle projection $\pi : B \to M$. We assign to $\Gamma^\infty(M \leftarrow B)$ the induced topology with respect to the $\mathfrak{D}_F$-topology of $C^\infty(M, B)$. Define the relative support of a section ψ with respect to another φ by
$$\mathrm{supp}_\psi(\varphi) = \{x \in M : \varphi(x) \neq \psi(x)\}, \tag{3}$$
then we refine it so that the subsets
$$\mathcal{V}_\varphi = \{\psi \in \Gamma^\infty(M \leftarrow B) : \mathrm{supp}_\varphi(\psi) \subset M \text{ is compact}\} \tag{4}$$
are open. Next we have to choose a candidate for the modelling LCS of 2.2. For this we proceed as in the proof of Theorem 42.1 in [KM97], we write the chart mapping first and then choose the appropriate LCS. Define
$$u_\varphi : \mathcal{U}_\varphi \subset \mathcal{V}_\varphi \to E_\varphi \ , \quad \psi \mapsto \exp_\varphi^{-1}(\psi) \ , \tag{5}$$
where exp is the exponential mapping associated to some Riemannian metric on B. In particular
$$u_\varphi(\psi)(x) \equiv \exp^{-1}_{\varphi(x)}(\psi(x)) \in V_{\varphi(x)}B \ ,$$
where $V_{\varphi(x)}B$ is the vertical subspace of tangent space of B at $\varphi(x)$. Now, both $\varphi(x)$, $\psi(x)$ belong to the same fiber $\pi^{-1}(x)$, hence the vector selected by exp will necessarily be vertical, moreover the set of points in M where ψ differs from φ is compact, hence $u_\varphi(\psi)$ will vanish identically outside some compact region. Therefore E_φ is naturally identified with the space $\Gamma^\infty_c(M \leftarrow \varphi^* VB)$ of compactly supported sections of the (vector) *pull-back* bundle of $VB \to B$ along $\varphi : M \to B$. The latter is a LCS with the final topology induced by
$$\varinjlim_{K \subset M} \Gamma^\infty_K(M \leftarrow \varphi^* VB) = \Gamma^\infty_c(M \leftarrow \varphi^* VB) \tag{6}$$
where each $\Gamma^\infty_K(M \leftarrow \varphi^* VB)$ is equipped with the topology of uniform convergence over the compact $K \subset M$ for all derivatives.

Theorem 2.3 (Proposition 4.8 pp. 38 [Mic80]). *Let (E, π, M) be a finite dimensional vector bundle, then $\Gamma^\infty_c(M \leftarrow E) \subset \Gamma^\infty(M \leftarrow E)$, endowed with the limit Fréchet topology of* (6)*, is the maximal LCS contained in $\Gamma^\infty(M \leftarrow E)$. Moreover it is a complete, nuclear and Lindelöf space, hence paracompact and normal.*

In the sequel, we will also make use of topological duals of our modelling vector spaces, notationally we write
$$(\Gamma^\infty_c(M \leftarrow \varphi^* VB))' \equiv \Gamma^{-\infty}(M \leftarrow \varphi^* VB), \tag{7}$$
elements of this space will be called distributional sections. This space is the dual of a LF space and carries the same topology used for distributional sections. From time to time we will take advantage of this fact which allows us to write derivatives of functionals, formally, as integrals using the Schwartz kernel theorem.

We stress that the choice of the chart $(\mathcal{U}_\varphi, u_\varphi)$ has to be made with care: (5) has to be well defined.[1] Then $\Gamma^\infty(M \leftarrow B)$ becomes a smooth manifolds, by Theorem 8.6 pp.78 in [Mic80], transition mappings are diffeomorphisms according to Definition 2.1, furthermore the smooth structure induced by $(\mathcal{U}_\varphi, u_\varphi)_\varphi \in \Gamma^\infty(M \leftarrow B)$ is independent from the choice of Riemannian metric h on B. We call the chart constructed above Γ^∞-*local charts* in order to separate them from the local chart of finite dimensional manifold that were mentioned before.

[1] By construction of $\mathcal{V}_\varphi$, that is always possible provided we identify $u_\varphi(\mathcal{U}_\varphi) \subset \Gamma^\infty_c(M \leftarrow \varphi^* VB)$ with a neighborhood of the zero section small enough that exp is a diffeomorphism.

The tangent space at each point φ is $T_\varphi\Gamma^\infty(M \leftarrow B) \equiv \Gamma_c^\infty(M \leftarrow \varphi^*VB)$. The (kinematic) tangent bundle $(T\Gamma^\infty(M \leftarrow B), \tau_\Gamma, \Gamma^\infty(M \leftarrow B))$ is defined in analogy with the finite dimensional case, and carries a canonical infinite dimensional bundle structure with trivializations

$$t_\varphi : \tau_\Gamma^{-1}(\mathcal{U}_\varphi) \to \mathcal{U}_\varphi \times \Gamma_c^\infty(M \leftarrow \varphi^*B).$$

With those mappings we can identify points of $T\Gamma^\infty(M \leftarrow B)$ by $t_\varphi^{-1}\left(\varphi, \vec{X}_\varphi\right)$. With those trivializations a tangent vector to $\Gamma^\infty(M \leftarrow B)$, i.e. an element of $T_\varphi\Gamma^\infty(M \leftarrow B)$, can equivalently be seen as a section of the vector bundle $\Gamma_c^\infty(M \leftarrow \varphi^*B)$. When using the latter interpretation, we will write the section in local coordinates as $\vec{X}(x) = \vec{X}^i(x)\partial_i\big|_{\varphi(x)}$. Finally we will use Roman letters, e.g. $(\vec{s}, \vec{u}, \dots)$ to denote elements of the dual space $\Gamma^{-\infty}(M \leftarrow \varphi^*B) \equiv \left(\Gamma_c^\infty(M \leftarrow \varphi^*B)\right)'$.

Definition 2.4. A connection over the (possibly infinite dimensional) bundle (B, π, X) is a vector-valued one form $\Phi \in \Omega^1(B; VB)$ satisfying

(*i*) $\mathrm{Im}(\Phi) = VB$,
(*ii*) $\Phi \circ \Phi = \Phi$.

Given a connection it is always possible to associate its canonical Christoffel form $\Gamma \doteq id_{TB} - \Phi$. In our particular case with $B = T\Gamma^\infty(M \leftarrow B)$, has canonical trivialization

$$Tt_\varphi : \tau_{T\Gamma}^{-1} \circ \tau_\Gamma^{-1}(\mathcal{U}_\varphi) \to \mathcal{U}_\varphi \times \Gamma_c^\infty(M \leftarrow \varphi^{*V}B) \times \Gamma_c^\infty(M \leftarrow \varphi^*VB) \times \Gamma_c^\infty(M \leftarrow \varphi^*VB).$$

Therefore given $Tt_\varphi^{-1}\left(\vec{Y}_\varphi, \vec{S}_{\vec{X}}\right) \in TT\Gamma^\infty(M \leftarrow B)$, we can write the connection locally as

$$t_\varphi^*\Phi\left(\vec{Y}_\varphi, \vec{S}_{\vec{X}}\right) = \left(\vec{0}_\varphi, \vec{S}_{\vec{X}} - \Gamma_\varphi(\vec{X}, \vec{Y})\right),$$

where

$$\Gamma_\varphi : \Gamma_c^\infty(M \leftarrow \varphi^*VB) \times \Gamma_c^\infty(M \leftarrow \varphi^*VB) \to \Gamma_c^\infty(M \leftarrow \varphi^*VB)$$

can be chosen to be linear in the first two entries. For additional details about connections see [KM97] Chapter VI, section 37.

3. Observables

By an observable, equivalently a functional, we mean a smooth mapping

$$F : \mathcal{U} \subset \Gamma^\infty(M \leftarrow B) \to \mathbb{R},$$

where $\mathcal{U}$ is an open set in the CO-topology introduced above. Since smoothness is a Γ^∞-local issue, a functional F is smooth if and only if, given any family of Γ^∞-local charts $\{\mathcal{U}_\varphi, u_\varphi\}_{\varphi\in\mathcal{U}}$ its localization

$$F_\varphi \doteq F \circ u_\varphi^{-1} : \Gamma_c^\infty(M \leftarrow \varphi^*VB) \to \mathbb{R}, \tag{8}$$

is smooth in the sense of Definition 2.1.

The first notion we introduce is the spacetime support of a functional. The idea is to follow the definition of support given in [BFR19], and account for the lack of linear structure on the fibers of the configuration bundle B.

Definition 3.1. Let F be a functional over $\mathcal{U}$, CO-open, then its support is the closure in M of the subset $\{x \in M$ s.t. $\forall\ V \subset M$ open neighborhood of x, $\exists\varphi \in \mathcal{U}$, $\vec{X}_\varphi \in \Gamma_c^\infty(M \leftarrow \varphi^*VB)$ having $\mathrm{supp}(\vec{X}_\varphi) \subset V$, for which $F_\varphi(\vec{X}_\varphi) = F_\varphi(0)\}$. The set of observables over $\mathcal{U}$ with compact spacetime support will be denoted by $\mathcal{F}_c(B,\mathcal{U})$.

Let us display some examples of functionals. Given $\alpha \in C^\infty(B, \mathbb{R})$, consider

$$F_\alpha : \Gamma^\infty(M \leftarrow B) \to \mathbb{R} : \varphi \mapsto F_\alpha(\varphi) \doteq \begin{cases} \frac{1}{1+\sup_M(\alpha(\varphi))} & \alpha(\varphi) \text{ bounded },\\ 0 & \text{otherwise }. \end{cases} \tag{9}$$

If $f \in C_c^\infty(M)$ and $\lambda \in \Omega_m(J^rB)$ define

$$\mathcal{L}_{f,\lambda} : \Gamma^\infty(M \leftarrow B) \to \mathbb{R} : \varphi \mapsto \mathcal{L}_{f,\lambda}(\varphi) \doteq \int_M f(x) j^r\varphi^*\lambda(x) \mathrm{d}\mu_g(x). \tag{10}$$

On the other hand if f, λ are as above and $\chi : \mathbb{R} \to \mathbb{R}$ with $0 \le \chi \le 1$, $\chi(t) = 1\ \forall|t| \le 1/2$ and $\chi(t) = 0\ \forall|t| \ge 1/2$ define

$$G_{f,\lambda,\chi} : \Gamma^\infty(M \leftarrow B) \to \mathbb{C} : \varphi \mapsto G_{f,\lambda,\chi}(\varphi) \doteq e^{1-\chi\left((\mathcal{L}_{f,\lambda}(\varphi))^2\right)}. \tag{11}$$

We can endow $\mathcal{F}_c(B,\mathcal{U})$ with the following operations

1) $(F,G) \mapsto (F+G)(\varphi) \doteq F(\varphi)+G(\varphi)$;
2) $(z \in \mathbb{C}, F) \mapsto (zF)(\varphi) \doteq zF(\varphi)$;
3) $(F,G) \mapsto (F \cdot G)(\varphi) \doteq F(\varphi)G(\varphi)$;
4) $F \mapsto F^*$, with $F^*(\varphi) \doteq \overline{F(\varphi)}$.[2]

It can be shown that those operation preserve the compactness of the support, turning $\mathcal{F}_c(B,\mathcal{U})$ into a commutative **-algebra with unity* where the unit element is given by $\varphi \mapsto 1 \in \mathbb{R}$. That involution and scalar multiplication are support preserving is trivial, to see that for multiplication and sum we use

Lemma 3.2. *Let F, G be functionals over $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ CO-open subset, then*

(i) $\mathrm{supp}(F+G) \subset \mathrm{supp}(F) \cup \mathrm{supp}(G)$,
(ii) $\mathrm{supp}(F \cdot G) \subset \mathrm{supp}(F) \cup \mathrm{supp}(G)$.

Before the proof we note that the more restrictive version of (ii) with the intersection of domains does not hold in general, this can be checked by taking a constant functional $G(\varphi) \equiv c \; \forall \varphi \in \mathcal{U}$, then $\mathrm{supp}(G) = \emptyset$ while $\mathrm{supp}(F+G)$, $\mathrm{supp}(F \cdot G) = \mathrm{supp}(F)$.

Proof. Suppose that $x \notin \mathrm{supp}(F) \cup \mathrm{supp}(G)$, then there is an open neighborhood V of x such that for any $X \in \Gamma^\infty_c(M \leftarrow VB)$ with $\mathrm{supp}(X) \subset V$, and any $\varphi \in \mathcal{U}$ we have $(F+G)_\varphi(\vec{X}_\varphi) = F_\varphi(\vec{X}_\varphi) + G_\varphi(\vec{X}_\varphi) = F_\varphi(0) + G_\varphi(0)$, so $x \notin \mathrm{supp}(F+G)$. The other follows analogously. □

Using the notion of B-differentiability we can induce a related differentiability for functionals over $\Gamma^\infty(M \leftarrow B)$, in the same spirit as done for mappings between manifolds.

Definition 3.3. Let $\mathcal{U}$ be CO-open, a functional $F \in \mathcal{F}_c(B,\mathcal{U})$ is differentiable of order k at $\varphi \in \mathcal{U}$ if for all $0 \leq j \leq k$ the functionals $F^{(j)}_\varphi[0] : \otimes^j \left(\Gamma^\infty_c(M \leftarrow \varphi^*VB)\right) \to \mathbb{R} : (\vec{X}_1, \ldots, \vec{X}_j) \mapsto F^{(j)}_\varphi[0](\vec{X}_1, \ldots, \vec{X}_j)$ are linear and continuous with

$$\begin{aligned} F^{(j)}_\varphi[u_\varphi(\varphi)](\vec{X}_1, \ldots, \vec{X}_j) &\doteq \left.\frac{d^j}{dt_1 \ldots dt_j}\right|_{t_1=\ldots=t_j=0} F_\varphi(t_1\vec{X}_1 + \ldots + t_j\vec{X}_j) \\ &= \left\langle F^{(j)}_\varphi[0], \vec{X}_1 \otimes \ldots \otimes \vec{X}_j \right\rangle . \end{aligned}$$

If F is differentiable of order k at each $\varphi \in \mathcal{U}$ we say that F is differentiable of order k in $\mathcal{U}$. Whenever F is differentiable of order k in $\mathcal{U}$ for all $k \in \mathbb{N}$ we say that F is smooth and denote the set of smooth functionals as $\mathcal{F}_0(B,\mathcal{U})$.

When F is smooth the condition of Definition 3.3 is independent from the chart we use to evaluate the B differential: suppose we take charts $(\mathcal{U}_\varphi, u_\varphi)$, $(\mathcal{U}_\psi, u_\psi)$ with $\varphi \in \mathcal{U}_\psi$, then by Faà di Bruno's formula

$$\begin{aligned} & F^{(j)}_\psi[u_\psi(\varphi)](\vec{X}_1, \ldots, \vec{X}_j) \\ &= \sum_{\pi \in \mathscr{P}(\{1,\ldots,j\})} F^{(|\pi|)}_\varphi[0] \left(d^{(|I_1|)} u_{\varphi\psi}[u_\psi(\varphi)]\Big(\bigotimes_{i \in I_1} \vec{X}_i\Big), \ldots, d^{(|I_{|\pi|}|)} u_{\varphi\psi}[u_\psi(\varphi)]\Big(\bigotimes_{i' \in I_{|\pi|}} \vec{X}_{i'}\Big) \right), \end{aligned} \tag{12}$$

where π is a partition of $\{1, \ldots, j\}$ into $|\pi|$ smaller subsets $I_1, \ldots, I_{|\pi|}$ and we denoted by $u_{\varphi\psi}$ the transition function $u_\varphi \circ u_\psi^{-1}$. We immediately see that the right hand side is B-smooth by smoothness of the transition function, therefore the left hand side ought to be B-smooth as well. Incidentally the same kind of reasoning shows Definition 3.3 is independent from the Γ^∞-local atlas used for practical calculations.

Although this is enough to ensure B-differentiability, in the sequel we shall introduce a connection on the bundle $T\Gamma^\infty(M \leftarrow B) \to \Gamma^\infty(M \leftarrow B)$ so that (12) can be written as an equivalence between two single terms involving the covariant derivatives. In particular we will choose a linear connection, that is a $T\Gamma^\infty(M \leftarrow B)$-valued differential one form: $\Phi \in \Omega^1\big(\Gamma^\infty(M \leftarrow B); T\Gamma^\infty(M \leftarrow B)\big)$. Given a point $(\varphi, X) \in T\Gamma^\infty(M \leftarrow B)$ and an element in the fiber of $(\varphi, \vec{X})$, say $T\tilde{u}_\varphi^{-1}(\vec{Y}_\varphi, \vec{S}_{\vec{X}}) \in TT\Gamma^\infty(M \leftarrow B)$, the action of the connection is defined by

$$T\tilde{u}_\varphi \circ \Phi \circ T\tilde{u}_\varphi^{-1}(\vec{Y}_\varphi, \vec{S}_{\vec{X}}) \doteq \left(\vec{0}_\varphi, \vec{S}_{\vec{X}} - \Gamma_\varphi(\vec{X}, \vec{Y})\right), \tag{13}$$

where $\tilde{u}_\varphi \equiv Tu_\varphi : T\Gamma^\infty(M \leftarrow B)|_{\tilde{\mathcal{U}}_\varphi} \to \mathcal{U}_\varphi \times \Gamma^\infty_c(M \leftarrow \varphi^*VB)$ are the charts of the tangent bundle $T\Gamma^\infty(M \leftarrow B)$. One can always define this infinite dimensional connection as follows: consider any connection Γ on the

[2] In adherence to standard clFT, we use real functionals, which makes involution a trivial operation; we remark though that one could repeat *mutatis mutandis* everything with $\mathbb{R}$ replaced by $\mathbb{C}$, then involution is not trivial anymore.

typical fiber F of the bundle B, the latter will induce a linear connection $\varphi^*\Gamma$ on the vector bundle $M \leftarrow \varphi^* VB$, which defines the action of

$$\begin{aligned}\Gamma_\varphi :&\Gamma^\infty_c(M \leftarrow \varphi^*VB) \times \Gamma^\infty_c(M \leftarrow \varphi^*VB) \to \Gamma^\infty_c(M \leftarrow \varphi^*VB)\\ &(\vec{X},\vec{Y}) \mapsto \Gamma_\varphi(\vec{X}_\varphi,\vec{Y}_\varphi)\end{aligned}$$

by

$$\Gamma_\varphi(\vec{X},\vec{Y})(x) = \Gamma(\varphi(x))^i_{jk}\vec{X}^j(\varphi(x))\vec{Y}^k(\varphi(x))\partial_i\big|_{\varphi(x)};$$

where $X^j\partial_j\big|_\varphi$, $\vec{Y}^k_\varphi\partial_k\big|_\varphi$ are the expressions in local coordinates of $\vec{X}$, $\vec{Y} \in \Gamma^\infty_c(M \leftarrow \varphi^*VB)$. One can show, using smoothness of the coefficients of the pull-back connection $\varphi^*\Gamma$ and the Ω-lemma pp. 80 section 8.7 in [Mic80], that this is a well defined B-smooth connection. Armed with (13) we can define the notion of covariant differential setting recursively

$$\begin{aligned}\nabla^{(1)}F_\varphi[0](\vec{X}) &\doteq F^{(1)}_\varphi(\vec{X}),\\ \nabla^{(n)}F_\varphi[0](\vec{X}_1,\ldots,\vec{X}_n) &\doteq F^{(n)}_\varphi(\vec{X}_1,\ldots,\vec{X}_n)\\ &\quad+\sum_{j=1}^n\frac{1}{n!}\sum_{\sigma\in\mathcal{P}(n)}\nabla^{(n-1)}F_\varphi(\Gamma_\varphi(\vec{X}_{\sigma(j)},\vec{X}_{\sigma(n)}),\vec{X}_{\sigma(1)},\ldots,\widehat{\vec{X}_{\sigma(j)}},\ldots\vec{X}_{\sigma(n-1)}),\end{aligned}\tag{14}$$

where $\mathcal{P}(n)$ denotes the set of permutation of n elements. In this way we can extend properties of iterated derivatives, which are locally defined, globally. The price we pay is that, a priori, the property might depend on the connection chosen.

Lemma 3.4. *Let $\mathcal{U}$ be a locally convex, CO-open subset, and $F : \mathcal{U} \to \mathbb{R}$ a differentiable functional of order one, then*

$$\mathrm{supp}(F) = \overline{\bigcup_{\varphi\in\mathcal{U}}\mathrm{supp}\left(F^{(1)}_\varphi[0]\right)}$$

Proof. Suppose that $x \in \mathrm{supp}(F)$, then by definition for all open neighborhoods V of x there is $\varphi \in \mathcal{U}$ and $\vec{X}_\varphi \in \Gamma^\infty_c(M \leftarrow \varphi^*VB)$ with $\mathrm{supp}(\vec{X}) \subset V$ having $F_\varphi(\vec{X}_\varphi) \neq F_\varphi(0)$, using the convexity of $\mathcal{U}$ and the fundamental theorem of calculus we obtain that

$$F_\varphi(\vec{X}_\varphi) - F_\varphi(0) = \int_0^1 F^{(1)}_\varphi[\lambda\vec{X}_\varphi](\vec{X}_\varphi)\mathrm{d}\lambda \neq 0.$$

Thus for at least for some $\lambda_0 \in (0,1)$, the integrand is not zero, setting $\psi = u^{-1}_\varphi(\lambda_0\vec{X}_\varphi)$, we obtain

$$F^{(1)}_\psi[0]\left(d^{(1)}u_{\varphi\psi}[\lambda_0\vec{X}_\varphi](\vec{X}_\varphi)\right) \neq 0.$$

On the other hand if $x \in \mathrm{supp}\left(F^{(1)}_\varphi[0]\right)$ for some $\varphi \in \mathcal{U}$, then there is $\vec{X}_\varphi \in \Gamma^\infty_c(M \leftarrow \varphi^*VB)$ having $\vec{X}_\varphi(x) \neq \vec{0}$ for which $F^{(1)}_\varphi[0](\vec{X}_\varphi) \neq 0$, as a result, define

$$F_\varphi(\epsilon\vec{X}_\varphi) = F_\varphi(0) + \int_0^\epsilon F^{(1)}_\varphi[\lambda\vec{X}_\varphi](\vec{X}_\varphi)\mathrm{d}\lambda$$

having chosen ϵ small enough so that the integral is not vanishing. □

Definition 3.5. Let $\mathcal{U}$ be CO-open. The subspaces of $\mathcal{F}_c(B,\mathcal{U})$ defined by

- the set of smooth functionals $F \in \mathcal{F}_c(B,\mathcal{U})$ such that for each $\varphi \in \mathcal{U}$, the integral kernel associated to $\nabla^{(k)}F_\varphi[0]$,
$$\nabla^{(k)}F_\varphi[0](\vec{X}_1,\ldots,\vec{X}_k) = \int_{M^k}\nabla^{(k)}f_\varphi[0](x_1,\ldots,x_k)\vec{X}_1(x_1)\cdots\vec{X}_k(x_k)d\mu_g(x_1,\ldots,x_k)$$
has $\nabla^{(k)}f_\varphi[0] \in \Gamma^\infty_c\big(M \leftarrow \boxtimes^k\big(\varphi^*VB'\big)\big)$. We denote this set by $\mathcal{F}_{\mathrm{reg}}(B,\mathcal{U})$.
- the set of smooth functionals $F \in \mathcal{F}_c(B,\mathcal{U})$ such that for each $\varphi \in \mathcal{U}$, $\mathrm{supp}\big(\nabla^{(2)}F_\varphi[0]\big) \subset \triangle_2(M)$, the latter being the diagonal of $M \times M$. We denote this set by $\mathcal{F}_{loc}(B,\mathcal{U})$.
- the set of $F \in \mathcal{F}_{loc}(B,\mathcal{U})$ such that for each $\varphi \in \mathcal{U}$, the integral kernel associated to $\nabla^{(1)}F_\varphi[0] \equiv F^{(1)}_\varphi[0]$ has $f^{(1)}_\varphi[0] \in \Gamma^\infty\big(M \leftarrow (\varphi^*VB)'\big)$. We denote this set by $\mathcal{F}_{\mu loc}(B,\mathcal{U})$.

These three sets are respectively called the space of regular, local and microlocal functionals.

Using the Schwartz kernel theorem, we can equivalently define microlocal functionals by requiring $\{F \in \mathcal{F}_{loc}(B,\mathcal{U}) : \mathrm{WF}\big(F^{(1)}_\varphi[0]\big) = \emptyset \ \forall \varphi \in \mathcal{U}\}$. Other authors add also further requirements, for example in [Bro+18], microlocal functionals have the additional property that given any $\varphi \in \mathcal{U}$ there exists an open neighborhood $\mathcal{V} \ni \varphi$ in which $f^{(1)}_{\varphi'}[0] \in \Gamma^\infty\left(M \leftarrow (\varphi^* VB)'\right)$ depends on the kth order jet of φ' for all $\varphi' \in \mathcal{V}$ and some $k \in \mathbb{N}$. We choose to give a somewhat more general description which however will turn out to be almost equivalent by Proposition 3.9. Finally we stress that the definition of local functionals together with Lemma 3.4 shows that that $\mathrm{supp}\big(\nabla^{(k)} F_\varphi[0]\big) \subset \triangle_k(M)$ for each $k \in \mathbb{N}$.

As remarked earlier, writing differentials with a connection, does yield a definition which is independent from the Γ^∞-local chart chosen to perform the calculations, however, we have to check that Definition 3.5 is independent from the connection chosen to perform the calculation. Suppose therefore we have connections Φ, $\widehat{\Phi}$, which induce covariant derivatives ∇, $\widehat{\nabla}$, and $F \in \mathcal{F}_c(B,\mathcal{U})$ is local with respect to the second connection, then we have

$$\left(\nabla^{(2)} F_\varphi - \widehat{\nabla}^{(2)} F_\varphi\right)[0](\vec{X}_1, \vec{X}_2) = F^{(1)}_\varphi[0]\left(\Gamma_\varphi(\vec{X}_1, \vec{X}_2) - \widehat{\Gamma}_\varphi(\vec{X}_1, \vec{X}_2)\right).$$

Due to linearity of the connection in both arguments, when the two sections $\vec{X}_1$, $\vec{X}_2$ have disjoint support the resulting vector field is identically zero, so that by linearity of $F^{(1)}_\varphi[0](\cdot)$ the expression is zero and locality is preserved. As a result, since $\nabla^{(1)} F_\varphi[0] \equiv F^{(1)}_\varphi[0]$, we immediately obtain that microlocality is independent as well. Regular functionals do not depend on the connection used to perform calculations either: this is easily seen by induction. If $k = 1$ this is trivial since $\nabla^{(1)} F \equiv F^{(1)}$, for arbitrary k one simply notes that $\left(\nabla^{k)} F_\varphi - \widehat{\nabla}^{(k)} F_\varphi\right)[0](\ldots)$ depends on terms of order $l \leq k-1$ and applies the induction hypotheses. We stress that in particular cases, such as when $B = M \times \mathbb{R}$, $TC^\infty(M) \equiv C^\infty(M) \times C^\infty_c(M)$, which is integrable and we are allowed to choose a trivial connection, in which case the differential and the covariant derivative coincide.

It is also possible to formulate Definition 3.5 in terms of differentials instead of covariant derivative, then the above argument can be used again to show that regular and local functionals do not depend on the choice of the chart.

When dealing with microlocal functionals we will sometimes use the following notation coming from application of Schwartz integral kernel theorem:

$$F^{(1)}_\varphi[0](\vec{X}_\varphi) = \int_M f^{(1)}_\varphi[0](\vec{X}_\varphi)(x) d\mu_g(x) = \int_M f^{(1)}_\varphi[0]_i(x) X^i_\varphi(x) d\mu_g(x), \tag{15}$$

where repeated indices denotes summation of vector components as usual with Einstein notation and $X^i_\varphi(x) \in \varphi^* p^{-1}(x)$ denotes the component of the section along the typical fiber of the vector bundle $(\Gamma^\infty_c(M \leftarrow \varphi^* VB), \varphi^* p, M)$.

If we go back to the examples of functionals given earlier we find that (9) does not belong to any class, while (11) is a regular functional that however fails to be local. If $D \subset M$ is a compact subset and χ_D its characteristic function then

$$\varphi \mapsto \mathcal{L}_{\chi_D,\lambda}(\varphi) \doteq \int_M \chi_D(x) \lambda(j^r\varphi)(x) \mathrm{d}\mu_g(x).$$

is a local functional which however is not microlocal due to the sigularities localized in the boundary of D. Finally we claim that (10) is a microlocal functional. To see it, let us consider a particular example where

$$\mathcal{L}_{f,\lambda}(\varphi) = \int_M f j^1\varphi^* \lambda = \int_M f(x) \lambda(j^r\varphi)(x) d\mu_g(x)$$

taking the first derivative and integrating by parts yields

$$\mathcal{L}^{(1)}_{f,\lambda,\varphi}[0](\vec{X}_\varphi) = \int_M f(x) \left\{\frac{\partial \lambda}{\partial y^i} - d_\mu\left(\frac{\partial \lambda}{\partial y^i_\mu}\right)\right\}(x) X^i_\varphi(x) \mathrm{d}\mu_g(x), \tag{16}$$

setting

$$\lambda^{(1)}_{f,\varphi}[0](x) \doteq f(x) \left\{\frac{\partial \lambda}{\partial y^i} - d_\mu\left(\frac{\partial \lambda}{\partial y^i_\mu}\right)\right\}(x) dy^i \wedge \mathrm{d}\mu_g(x), \tag{17}$$

we see that the integral kernel of the first derivative in φ of (10), $\lambda^{(1)}_{f,\varphi}[0]$, belongs to $\Gamma^\infty\left(M \leftarrow (\varphi^* VB)'\right)$. This last example is important because it shows that functionals obtained by integration of pull-backs of m-forms λ are essentially microlocal. One could ask whether the converse can hold, i.e. if all microlocal functionals have this form; the answer will be given in Proposition 3.9. We now give characterizations for locality and microlocality.

Definition 3.6. Let $\mathcal{U}$ be CO-open, a functional $F \in \mathcal{F}_c(B,\mathcal{U})$ is called:

(*i*) φ_0-additive if for all $\varphi_j \in \mathcal{U}_{\varphi_0} \cap \mathcal{U}$ having $\mathrm{supp}_{\varphi_0}(\varphi_1) \cap \mathrm{supp}_{\varphi_0}(\varphi_{-1}) = \emptyset$, setting $\vec{X}_j = u_{\varphi_0}(\varphi_j)$, $j = 1, -1$ and supposing that $\vec{X}_1 + \vec{X}_{-1} \in u_{\varphi_0}(\mathcal{U}_{\varphi_0} \cap \mathcal{U})$, we have

$$F_{\varphi_0}(\vec{X}_1+\vec{X}_{-1}) = F_{\varphi_0}(\vec{X}_1) - F_{\varphi_0}(0) + F_{\varphi_0}(\vec{X}_{-1}). \tag{18}$$

(*ii*) additive if for all $\varphi_j \in \mathcal{U}$, $j = 1, 0, -1$, with $\mathrm{supp}_{\varphi_0}(\varphi_1) \cap \mathrm{supp}_{\varphi_0}(\varphi_{-1}) = \emptyset$, setting

$$\varphi = \begin{cases} \varphi_1 & \text{in } \mathrm{supp}_{\varphi_0}(\varphi_{-1})^c \\ \varphi_{-1} & \text{in } \mathrm{supp}_{\varphi_0}(\varphi_1)^c \end{cases}$$

we have

$$F(\varphi) = F(\varphi_1) + F(\varphi_0) - F(\varphi_{-1}). \tag{19}$$

We remark that (*ii*) is equivalent to the definition of additivity present in [BFR16]. Before the proof of the equivalence of those two relations, we prove a technical lemma.

Lemma 3.7. *Let φ_1, φ_0, $\varphi_{-1} \in \Gamma^\infty(M \leftarrow B)$ have $\mathrm{supp}_{\varphi_0}(\varphi_1) \cap \mathrm{supp}_{\varphi_0}(\varphi_{-1}) = \emptyset$, then there exist $n \in \mathbb{N}$, a finite family of sections*

$$\left\{\varphi_{(k,l)}\right\}_{k,l \in \{1,\dots,n\}} \tag{20}$$

for which the following conditions holds:

(*a*) *For each k, $l \in \mathbb{N}$*

$$\left\{\varphi_{(k-1,l-1)}, \varphi_{(k,l-1)}, \varphi_{(k-1,l)}, \varphi_{(k+1,l)}, \varphi_{(k,l+1)}, \varphi_{(k+1,l+1)}\right\} \in \mathcal{U}_{\varphi_{(k,l)}}, \tag{21}$$

(*b*) *Moreover for each k, $l \in \mathbb{N}$ we can define sections $\{\vec{X}_k\}$, $\{\vec{Y}_l\} \in \Gamma^\infty_c(M \leftarrow \cdot^*VB)$, where*

$$\vec{X}_k \doteq \exp^{-1}_{\varphi_{(k-1,l)}}\left(\varphi_{(k,l)}\right), \tag{22}$$

$$\vec{Y}_l \doteq \exp^{-1}_{\varphi_{(k,l-1)}}\left(\varphi_{(k,l)}\right); \tag{23}$$

whose exponential flows generate all the above sections:

$$\varphi_1 = \exp\left(\vec{X}_n\right) \circ \cdots \circ \exp\left(\vec{X}_1\right) \circ \varphi_0 \equiv \varphi_{(n,0)};$$

$$\varphi_{-1} = \exp\left(\vec{Y}_n\right) \circ \cdots \circ \exp\left(\vec{Y}_1\right) \circ \varphi_0 \equiv \varphi_{(0,n)};$$

and

$$\varphi = \exp\left(\vec{X}_n\right) \circ \cdots \circ \exp\left(\vec{X}_1\right) \circ \exp\left(\vec{Y}_n\right) \circ \cdots \circ \exp\left(\vec{Y}_1\right) \circ \varphi_0.$$

Proof. Ideally we are taking φ_0 as a reference section, then application of a number of exponential flows of the above fields will generate new sections interpolating between φ_0 and $\varphi, \varphi_1, \varphi_{-1}$, such that each section in the interpolation procedure has the adjacent sections in the same chart (as in (21)). This is, for a pair of generic sections, not trivial; however, due to the requirement of mutual compact support between sections, our case is special. Indeed, let K be any compact containing $\mathrm{supp}_{\varphi_0}(\varphi_1) \cup \mathrm{supp}_{\varphi_0}(\varphi_{-1})$. Since B is itself a paracompact manifold, it admits an exhaustion by compact subsets and a Riemannian metric compatible with the fibered structure. The exponential mapping of this metric will have a positive injective radius throughout any compact subset of B. Thus let H be any compact subset of B containing the bounded subset

$$\Big\{b \in \pi^{-1}(K) \subset B : \sup_{x\in K} d(\varphi_0(x), b) < 2\max\big(\sup_{x\in K} d(\varphi_0(x), \varphi_1(x)), \sup_{x\in K} d(\varphi_0(x), \varphi_{-1}(x))\big)\Big\},$$

where d is the distance induced by the metric chosen. Let $\delta > 0$ be the injective radius of the metric on the compact H. If $r = \max\big(\sup_{x\in K} d(\varphi_0(x), \varphi_1(x)), \sup_{x\in K} d(\varphi_0(x), \varphi_{-1}(x))\big)$ there will be some finite $n \in \mathbb{N}$ such that $n\delta < r < (n+1)\delta$, and thus we can select a finite family of sections $\left\{\varphi_{(k,l)}\right\}_{k,l=1,\dots,n}$ interpolating between $\varphi_0 = \varphi_{(0,0)}$ and $\varphi_1 = \varphi_{(n,0)}$, $\varphi_{-1} = \varphi_{(0,n)}$, $\varphi = \varphi_{(n,n)}$ such that

$$(|k-k'|-1)\frac{\delta}{2} + (|l-l'|-1)\frac{\delta}{2} < \sup_{x\in K} d\left(\varphi_{(k,l)}(x), \varphi_{(k',l')}(x)\right) < (|k-k'|)\frac{\delta}{2} + (|l-l'|)\frac{\delta}{2},$$

This property essentially ensure us that we are interpolating in the right direction, that is, as k (resp. l) grows new sections are nearer to φ_1 (resp. φ_{-1}) and further away from φ_0. Setting

$$\vec{X}_{(k,l)} \doteq \exp^{-1}_{\varphi_{(k-1,l)}}\left(\varphi_{(k,l)}\right),$$

$$\vec{Y}_{(k,l)} \doteq \exp^{-1}_{\varphi_{(k,l-1)}}\left(\varphi_{(k,l)}\right),$$

we find the vector field interpolating between sections. Those vector fields are always well defined because, by construction, we choose adjacent sections to be separated by a distance where exp is still a diffeomorphism. Due to the mutual disjoint support of φ_1 and φ_{-1}, we can identify $\vec{X}_{(k,l)}$ (resp. $\vec{Y}_{(k,l)}$) with each other $\vec{X}_{(k,l')}$ (resp.

$\vec{Y}_{(k',l)}$), therefore it is justified to use one index to denote the vector fields as done in (22) and (23). Moreover, for each k, $l \in \mathbb{N}$, we have

$$\exp\left(\vec{X}_k\right) \circ \exp\left(\vec{Y}_l\right) = \exp\left(\vec{Y}_l\right) \circ \exp\left(\vec{X}_k\right);$$

which provides a well defined section φ. □

Proposition 3.8. *Let $F \in \mathcal{F}_0(B, \mathcal{U})$ then the following statements are equivalent:*

(1) *F is additive;*
(2) *F is φ_0-additive for all $\varphi_0 \in \mathcal{U}$;*
(3) *$F \in \mathcal{F}_{\mathrm{loc}}(B, \mathcal{U})$.*

Proof. Let us start proving the equivalence between (1) and (2).

(1) $\Rightarrow$ (2)**:** If $\varphi_j \in \mathcal{U} \cap \mathcal{U}_{\varphi_0}$ with $j = 1, 0, -1$ are as in (ii) above, take $\vec{X}_j$ such that $u^{-1}_{\varphi_0}(\vec{X}_j) = \varphi_j$. Writing (19) in terms of F_{φ_0} yields (18).

(2) $\Rightarrow$ (1)**:** Let us take sections φ_j with $j = 1, 0, -1$ such that $\mathrm{supp}_{\varphi_0}(\varphi_1) \cap \mathrm{supp}_{\varphi_0}(\varphi_{-1}) = \emptyset$, then we calculate $F(\varphi)$ combining Lemma 3.7 with φ-additivity for each section, yields

$$\begin{aligned}
F(\varphi) &= F_{\varphi_{(n-1,n-1)}}\left(\vec{X}_n + \vec{Y}_n\right) = F_{\varphi_{(n-1,n-1)}}\left(\vec{X}_n\right) + F_{\varphi_{(n-1,n-1)}}\left(\vec{Y}_n\right) - F_{\varphi_{(n-1,n-1)}}(0) \\
&= F\left(\varphi_{(n,n-2)}\right) + F\left(\varphi_{(n-1,n-1)}\right) - F\left(\varphi_{(n-1,n-2)}\right) \\
&\quad + F\left(\varphi_{(n-2,n)}\right) + F\left(\varphi_{(n-1,n-1)}\right) - F\left(\varphi_{(n-2,n-1)}\right) - F\left(\varphi_{(n-1,n-1)}\right) \\
&= F\left(\varphi_{(n,n-2)}\right) - F\left(\varphi_{(n-1,n-2)}\right) + F\left(\varphi_{(n-2,n)}\right) + F\left(\varphi_{(n-2,n-1)}\right) + F\left(\varphi_{(n-1,n-2)}\right) \\
&\quad - F\left(\varphi_{(n-2,n-2)}\right) - F\left(\varphi_{(n-2,n-1)}\right) \\
&= F\left(\varphi_{(n,n-2)}\right) + F\left(\varphi_{(n-2,n)}\right) - F\left(\varphi_{(n-2,n-2)}\right).
\end{aligned}$$

Repeating the above argument an extra $(n-2)$ times we arrive at

$$F(\varphi) = F\left(\varphi_{(n,0)}\right) + F\left(\varphi_{(0,n)}\right) - F\left(\varphi_{(0,0)}\right) \equiv F(\varphi_1) + F(\varphi_{-1}) - F(\varphi_0).$$

We conclude proving that (2) and (3) are equivalent.

(3) $\Rightarrow$ (2)**:** Take φ_j, $\vec{X}_j \doteq u_{\varphi_0}(\varphi_j)$, with $j = 1, 0, -1$ as in (i) Definition 3.6. Then

$$F_{\varphi_0}(\vec{X}_1 + \vec{X}_{-1}) - F_{\varphi_0}(\vec{X}_1) + F_{\varphi_0}(0) - F_{\varphi_0}(\vec{X}_{-1}) = \int_0^1 \frac{d}{dt}\Big(F_{\varphi_0}(\vec{X}_1 + t\vec{X}_{-1}) - F_{\varphi_0}(t\vec{X}_{-1})\Big)\, dt$$
$$= \int_0^1 \frac{d}{dt}\left(\int_0^1 \frac{d}{dh} F_{\varphi_0}(h\vec{X}_1 + t\vec{X}_{-1}) dh\right) dt = \int_0^1 \int_0^1 F^{(2)}_{\varphi_0}[h\vec{X}_1 + t\vec{X}_{-1}](\vec{X}_1, \vec{X}_{-1}) dh dt.$$

Now by locality we have that $\mathrm{supp}\left(F^{(2)}_{\varphi_0}\right) \subset \triangle_2 M$, however, $\mathrm{supp}(\vec{X}_1) \cap \mathrm{supp}(\vec{X}_{-1}) = \emptyset$ implying that the integrand on the right hand side of the above equation is identically zero.

(2) $\Rightarrow$ (3)**:** Fix any $\varphi_0 \in \mathcal{U}$, consider two vector fields $\vec{X}_1$, $\vec{X}_{-1} \in \Gamma^\infty\big(M \leftarrow \varphi_0^* VB\big)$ such that $\mathrm{supp}(\vec{X}_1) \cap \mathrm{supp}(\vec{X}_{-1}) = \emptyset$ and $\vec{X}_1 + \vec{X}_{-1} \in u_{\varphi_0}(\mathcal{U}_{\varphi_0})$, let also $\varphi_j \doteq u^{-1}_{\varphi_0}(\vec{X}_j)$ for $j = 1, -1$, then $\mathrm{supp}_\varphi(\varphi_1) \cap \mathrm{supp}_\varphi(\varphi_{-1}) = \emptyset$. By direct computation we get

$$\begin{aligned}
F^{(2)}_{\varphi_0}[0](\vec{X}_1, \vec{X}_{-1}) &= \left.\frac{d^2}{dt_1 dt_2}\right|_{t_1 = t_2 = 0} F_\varphi(t_1\vec{X}_1 + t_2\vec{X}_{-1}) \\
&= \left.\frac{d^2}{dt_1 dt_2}\right|_{t_1 = t_2 = 0} \Big(F_{\varphi_0}(t_1\vec{X}_1) - F_{\varphi_0}(0) + F_{\varphi_0}(t_2\vec{X}_{-1})\Big) \equiv 0,
\end{aligned}$$

which proves locality.

□

As a result, we have shown that locality and additivity are consistent concepts in a broader generality than done in [BFR19] of course additivity strongly relates to Bogoliubov's formula for S-matrices, therefore a priori we expect that whenever we can formulate the concept consistently in Definition 3.6, those must be equivalent formulations. We also mention that when the exponential map used to construct Γ^∞-local charts is a global diffeomorphism, then additivity and φ additivity becomes trivially equivalent since the chart open can be enlarged to $\mathcal{V}_\varphi \equiv \{\psi \in \Gamma^\infty(M \leftarrow B) : \mathrm{supp}_\varphi(\psi) \subset M \text{ is compact}\}$.

We stress that the Γ^∞-local notion of additivity i.e. (i) in Definition 3.6 is independent from the chart used, in fact suppose that F is φ_0-additive in $\{\mathcal{U}_{\varphi_0}, u_{\varphi_0}\}$, take another chart $\{\mathcal{U}'_{\varphi_0}, u'_{\varphi_0}\}$ such that $\mathcal{U}'_{\varphi_0} \cap \mathcal{U}_\varphi \neq \emptyset$, set $\vec{X}_j = u_{\varphi_0}(\varphi_j)$, $\vec{Y}_j = u'_{\varphi_0}(\varphi_j)$, for $j = 1, -1$, we have[3]

$$\begin{aligned}F \circ u'^{-1}_{\varphi_0}(\vec{Y}_1 + \vec{Y}_{-1}) &= F \circ u^{-1}_{\varphi_0} \circ u_{\varphi_0} \circ u'^{-1}_{\varphi_0}(\vec{Y}_1 + \vec{Y}_{-1}) = F \circ u^{-1}_{\varphi_0}(\vec{X}_1 + \vec{X}_{-1})\\ &= F \circ u^{-1}_{\varphi_0}(\vec{X}_1) - F \circ u^{-1}_{\varphi_0}(0) + F \circ u^{-1}_{\varphi_0}(\vec{X}_{-1})\\ &= F \circ u'^{-1}_{\varphi_0}(\vec{Y}_1) - F \circ u'^{-1}_{\varphi_0}(0) + F \circ u'^{-1}_{\varphi_0}(\vec{Y}_{-1})\end{aligned}$$

where $u_{\varphi_0} \circ u'^{-1}_{\varphi_0}(\vec{Y}_1 + \vec{Y}_{-1}) = \vec{X}_1 + \vec{X}_{-1}$ is due to the fact that the two vector fields have mutually disjoint supports. We then see that φ_0-additivity does not depend upon the chosen chart.

We now give the Γ^∞-local characterization of microlocality; we will find that, contrary to additivity, the latter representation will be limited to a chart domain, in the sense that the functional can be represented as an integral provided we shrink its domain to a chart, this representation however will not yield a global one, for, in general, it depends on the chart used.

Proposition 3.9. *Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open and $F \in \mathcal{F}_{\mu loc}(B, \mathcal{U})$, then $f^{(1)} : \mathcal{U} \subset \Gamma^\infty(M \leftarrow B) \ni \varphi \mapsto f^{(1)}_\varphi[0] \in \Gamma^\infty_c(M \leftarrow \varphi^* VB' \otimes \Lambda_m(M))$ is B-smooth if and only if for each $\mathcal{U}_{\varphi_0} \subset \mathcal{U}$ there is a m-form $\lambda_{F,\varphi_0} \equiv \lambda_{F,0}$ with $\lambda_{F,0}(j^\infty\varphi)$ having compact support for all $\varphi \in \mathcal{U}_{\varphi_0}$ such that*

$$F(\varphi) = F(\varphi_0) + \int_M (j^r_x\varphi)^* \lambda_{F,0}. \tag{24}$$

Proof. Suppose $F(\varphi) = F(\varphi_0) + \int_M (j^r\varphi)^*\lambda_{F,0}$ for all $\varphi \in \mathcal{U}_{\varphi_0}$, we evaluate $F^{(1)}_\varphi[0]$ and find that its integral kernel may always be recast in the form

$$f^{(1)}_\varphi[0](x) = e_i[\lambda, \varphi_0](j^{2r}_x\varphi) dy^i \otimes d\mu_g(x)$$

where $e_i[\lambda, F, \varphi_0] dy^i \otimes d\mu_g : \Gamma^\infty(M \leftarrow B) \to \Gamma^\infty_c(M \leftarrow \varphi^* VB' \otimes \Lambda_m(M))$ are the Euler-Lagrange equations associated to λ_{F,φ_0} evaluated at some field configuration. To show smoothness we can, being careful of using the Γ^∞-differential structure for the source space, apply the Ω-Lemma pp. 80 section 8.7 in [Mic80].

Conversely suppose that $f^{(1)}$ is B-smooth as a map, fix $\varphi_0 \in \mathcal{U}$ and call $\vec{X} = u_{\varphi_0}(\varphi)$, by microlocality combined with Schwartz kernel theorem

$$F(\varphi) - F(\varphi_0) = F_{\varphi_0}(\vec{X}) - F_{\varphi_0}(0) = \int_0^1 F^{(1)}_{\varphi_0}[t\vec{X}](\vec{X}) dt = \int_0^1 dt \int_M f^{(1)}_{\varphi_0}[t\vec{X}](\vec{X}).$$

Applying the Fubini-Tonelli theorem to exchange the integrals in the above relation yields our candidate for $j^r\varphi^*\lambda_{F,0}$: the m-form $x \mapsto \theta[\varphi](x) \equiv \int_0^1 f^{(1)}_{\varphi_0}[t\vec{X}](\vec{X})(x) dt$. We have to show that this element depends at most on $j^r_x\varphi$. Notice that, a priori, $\theta[\varphi](x)$ might not depend on $j^r_x\varphi$, however we can say that if φ_1, φ_2 agree on any neighborhood V of $x \in M$, then $\theta[\varphi_1]|_V = \theta[\varphi_2]|_V$. To see this, setting $\vec{X}_1 = u_{\varphi_0}(\varphi_1)$, $\vec{X}_2 = u_{\varphi_0}(\varphi_2)$ they agree in a suitably small neighborhood V' of x, moreover

$$\begin{aligned}\theta[\varphi_1](x) - \theta[\varphi_2](x) &= \int_0^1 dt \left(f^{(1)}_{\varphi_0}[t\vec{X}_1](\vec{X}_1)(x) - f^{(1)}_{\varphi_0}[t\vec{X}_2](\vec{X}_2)(x) \right)\\ &= \int_0^1 dt \left(f^{(1)}_{\varphi_0}[t\vec{X}_1]_i(x)\vec{X}^i_1(x) - f^{(1)}_{\varphi_0}[t\vec{X}_2]_i(x)\vec{X}^i_2(x) \right)\\ &= \int_0^1 dt \int_0^1 dh f^{(2)}_{\varphi_0}[t\vec{X}_2 + th\vec{X}_1 - th\vec{X}_2]_{ij}(x)(t\vec{X}_1 - t\vec{X}_2)^i(x)\vec{X}^j_1(x);\end{aligned}$$

where in the last equality we used locality of F and linearity of the derivative. The last line of the above equation identically vanishes in V' due the support properties of $f^{(2)}_{\varphi_0}$ and the fact that $\vec{X}_1|_{V'} = \vec{X}_2|_{V'}$. Therefore $\theta[\varphi](x) \in \varphi^* VB' \otimes \Lambda_m(M)$ depends at most on $\mathrm{germ}_x(\varphi)$ with $\varphi \in \mathcal{U}$.

We wish to apply the Peetre-Slovak theorem[4] to θ; the germ dependence hypotheses has been verified above, so one has to show that θ is also weakly regular, that is, if $\mathbb{R} \times M \ni (t, x) \mapsto \varphi_t(x) \in B$ is a jointly smooth mapping with $\varphi_t \in \Gamma^\infty(M \leftarrow B)$ for each $t \in \mathbb{R}$ such that there is a compact subset in M outside of which φ_t is constant for all $t \in \mathbb{R}$, then $(t, x) \mapsto \theta[\varphi_t](x)$ is again a jointly smooth compactly supported variation. Since θ is a compactly supported form, then it maps compactly supported variations into compactly supported variations. Suppose therefore that $\mathbb{R} \ni t \to \varphi_t \in \Gamma^\infty(M \leftarrow B)$ is a mapping as described above, then φ_t is a

[3]In the subsequent calculations we can assume, without loss of generality, that $\vec{Y}_1 + \vec{Y}_{-1} \in u'_{\varphi_0}(\mathcal{U}'_{\varphi_0})$, for if this is not the case we can use an argument involving Lemma 3.7 to make this expression meaningful.

[4]See *e.g.* Chapter 19 in [KMS13] for the precise statement of the theorem.

smooth mapping for the smooth structure of $\Gamma^\infty(M \leftarrow B)$ by Lemma 30.9 in [KM97]. By B-smoothness of the mapping θ, the composition

$$t \mapsto \theta[\varphi_t] \in \Gamma^\infty_c(M \leftarrow \varphi_t^* VB' \otimes \Lambda_m(M))$$

is a smooth curve as well. Also we can define a canonical fibered isomorphism $\varphi_t^* VB' \otimes \Lambda_m(M) \simeq \varphi_0^* VB' \otimes \Lambda_m(M)$ which, by Theorem 8.7 pp. 80 in [Mic80], induces a smooth mapping $\Gamma^\infty_c(M \leftarrow \varphi_t^* VB' \otimes \Lambda_m(M)) \simeq \Gamma^\infty_c(M \leftarrow \varphi^* VB' \otimes \Lambda_m(M))$. As a result we have that θ is a smooth mapping

$$t \mapsto \theta[\varphi_t] \in \Gamma^\infty_c(M \leftarrow \varphi_0^* VB' \otimes \Lambda_m(M)).$$

We infer by Lemma 30.9 in [KM97] that $\theta[\varphi] : M \times \mathbb{R} : (t,x) \mapsto \theta[\varphi_t](x) \in B$ is smooth. Note that the topology limit-Fréchet topology on $\Gamma^\infty_c(M \leftarrow \varphi_0^* VB' \otimes \Lambda_m(M))$ makes it a topological embedding both in $\Gamma^\infty(M \leftarrow \varphi_0^* VB' \otimes \Lambda_m(M))$ with the WO^∞ topology (Lemma 41.11 in [KM97]) and with the $\mathfrak{D}_F$ topology (4.11 pp 41. in [Mic80]). We can now apply the Peetre-Slovak theorem and deduce that for each neighborhood there exists $r \in \mathbb{N}$, an open neighborhood $U^r \subset J^r B$ of $j^r \varphi_0$ and a mapping $\lambda_{F,0} : J^r B \supset U^r \to \Gamma^\infty_c(M \leftarrow \varphi_0^* VB' \otimes \Lambda_m(M))$ such that $\lambda_{F,0}(j^r_x \varphi) = \theta[\varphi](x)$ for each φ with $j^r \varphi \in U^r$. Due to compactness of $\mathrm{supp}(\theta)$ we can take the order r to be uniform over M; then

$$F(\varphi) = F(\varphi_0) + \int_M \lambda_{F,0}(j^r_x \varphi).$$

$\square$

As mentioned above, this characterization is limited to the Γ^∞-local chart chosen: given charts $\{\mathcal{U}_{\varphi_j}, u_{\varphi_j}\}$, $j = 1, 2$ such that $\varphi \in \mathcal{U}_{\varphi_1} \cap \mathcal{U}_{\varphi_2}$, let $\vec{X}_j = u_{\varphi_j}(\varphi)$ and suppose F satisfies the hypothesis of Proposition 3.9, then according to (24)

$$F(\varphi) = F(\varphi_1) + \int_M (j^{r_1}\varphi)^* \lambda_{F,1} = F(\varphi_2) + \int_M (j^{r_2}\varphi)^* \lambda_{F,2}.$$

We can assume that $r_1 = r_2 \equiv r$, then using the same argument as in the proof of Proposition 3.9,

$$\begin{aligned}
(j^r \varphi)^* \lambda_{F,1}(x) &= \int_0^1 f^{(1)}_{\varphi_1}[t\vec{X}_1]\left(\vec{X}_1\right)(x)\mathrm{d}t \\
&= \int_0^1 f^{(1)}_{\varphi_2}[u_{\varphi_1\varphi_2}(t\vec{X}_1)]\left(u_{\varphi_1\varphi_2}(\vec{X}_1)\right)(x)\mathrm{d}t \\
&= \int_0^1 f^{(1)}_{\varphi_2}[u_{\varphi_1\varphi_2}(t\vec{X}_1)]\left(\vec{X}_2\right)(x)\mathrm{d}t,
\end{aligned}$$

whereas

$$(j^r \varphi)^* \lambda_{F,2}(x) = \int_0^1 f^{(1)}_{\varphi_2}[t\vec{X}_2]\left(\vec{X}_2\right)(x)\mathrm{d}t,$$

we therefore see that the lack of linearity of the transition mapping $u_{\varphi_1\varphi_2}$, namely $u_{\varphi_1\varphi_2}(t\vec{X}_1) \neq t u_{\varphi_1\varphi_2}(\vec{X}_1) = t\vec{X}_2$, does not allow us to conclude $(j^\infty\varphi)^* \lambda_{F,1}(x) = (j^\infty\varphi)^* \lambda_{F,2}(x)$.

We give here also another argument that prevents Γ^∞-local globality of the m forms obtained from Proposition 3.9. This relies on the variational sequence[5]: a cohomological sequence of forms over $J^r B$ for some finite $r \in \mathbb{N}$,

$$0 \longrightarrow \mathbb{R} \xrightarrow{E_0} \Omega^1(J^r B)/\sim \xrightarrow{E_1} \dots \longrightarrow \Omega^m(J^r B)/\sim \xrightarrow{E_m} \Omega^{m+1}(J^r B)/\sim \xrightarrow{E_{m+1}} \Omega^{m+2}_h(B)/\sim$$

$$\dots \longrightarrow \Omega^N(B) \xrightarrow{E_N} 0$$

where each element of the sequence is the quotient of the space of p-forms in $J^r B$ modulo some relation that cancel the exact forms (in the sense of the de-Rham differential on the manifold $J^r B$) and accounts for integration by parts when the order is greater then $m = \dim(M)$. In particular the mth differential E_m is the operator which, given a horizontal m-form, calculates its *Euler-Lagrange form* and the $(m+1)$th differential E_{m+1} is the operator which associates to each Euler-Lagrange form its *Helmholtz-Sonin form*. By the Poincaré lemma, if $\sigma \in \ker(E_{m+1})$, there exists a local chart (V^r, ψ^r) in $J^r B$ and a horizontal m-form $\lambda \in \Omega^m(V^r)$ having $E_m|_{V^r}(\lambda) = \sigma|_{V^r}$. Establishing whether this condition holds globally is the heart of the inverse problem in calculus of variation and can be formulated as follows: given equations satisfying some condition (the associated Helmholtz-Sonin form vanishes) do they arise from the variation of some Lagrangian? The variational sequence implies that this is always the case whenever the m-th cohomology group vanishes, therefore giving a sufficient conditions whenever some topological obstruction is not present.

[5] A complete exposition can be found in [Kru15].

Now, if Proposition 3.9 could somehow reproduce (24) for each $\varphi \in \mathcal{U}$ with an integral over the same m-form λ_F, then we would have found a way to circumvent the topological obstructions that ruin the exactness of the variational sequence. Furthermore in the derivation of $\lambda_{F,0}$ we did not even require that the associated Euler-Lagrange equations had vanishing Helmholtz-Sonin form, but instead a B-smoothness requirement that will always be met by integral functionals constructed from smooth geometric objects. It appears therefore that the two approaches bears some kind of duality: given a representative $f^{(1)}_\varphi[0] \in \Omega^{m+1}(J^rB)/\sim$ one can, on one hand, give a Γ^∞-*local Lagrangian* via Proposition 3.9 i.e. a global m-form on the bundle J^rB which however describe the functional only when evaluated in a small neighborhood of a reference section φ_0; on the other, prioritize Γ^∞-local globality, therefore having a local m-form defined on the bundle $J^r(\pi^{-1}U)$ for some open subset U of M, which however describe the functional for all sections of $\Gamma^\infty(U \leftarrow \pi^{-1}(U))$.

We encapsulate this observation in the following proposition.

Proposition 3.10. *Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open, $F \in \mathcal{F}_{\mu loc}(B, \mathcal{U})$ satisfying the hypotheses of Proposition 3.9. Fix $\varphi \in \mathcal{U}_{\varphi_0}$ and suppose that*

$$F(\varphi) = F(\varphi_0) + \int_M (j^\infty\varphi)^*\lambda_{F,0} = F(\varphi_0) + \int_M (j^\infty\varphi)^*\lambda'_{F,0}.$$

then $\lambda_{F,0} - \lambda'_{F,0} = d_h\theta$ for some $\theta \in \Omega^{m-1}_{\mathrm{hor}}(J^rB)$ if and only if the m-th de Rham cohomology group $H^m_{\mathrm{dR}}(B) = 0$. In particular the above condition is verified whenever B is a vector bundle with finite dimensional fiber and M is orientable non-compact and connected.

Proof. Using the notation introduced above for the variational sequence we have that $E_m(\lambda_{F,0}) = E_m(\lambda'_{F,0})$, since each of the two expressions equals $f^{(1)}_{\varphi_0}[0]$; thus their difference is zero and $\lambda_{f,\psi} - \lambda'_{f,\psi} \in \Omega^m(J^rB)/\sim$. The latter cohomology group is isomorphic, by the abstract de Rham Theorem, to $H^m_{\mathrm{dR}}(B)$, therefore $\lambda_{F,0} - \lambda'_{F,0} = d_h\theta$ if and only if $H^m_{\mathrm{dR}}(B) = 0$. When B is a vector bundle over M its de Rham cohomology group are isomorphic to those of M, which when orientable non-compact and connected, has $H^m_{\mathrm{dR}}(M) = 0$. The latter claim can be established using Poincaré duality, *i.e.* $H^m_{\mathrm{dR}}(M) \simeq H^0_{\mathrm{dR},c}(M)$, if M is non-compact and connected, *e.g.* when it is globally hyperbolic, there are no compactly supported functions with vanishing differential other then the zero function, so the m-th cohomology group is zero. □

We shall conclude this section by introducing generalized Lagrangians, which, as the name suggests, will be used to select a dynamic on $\Gamma^\infty(M \leftarrow B)$. We stress that unlike the usual notion of Lagrangian - either a horizontal m-form over J^rB or a morphism $J^rB \to \Lambda_m(M)$ - this definition will allow us to bypass all problems of convergence of integrals of forms in noncompact manifolds.

Definition 3.11. Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open. A generalized Lagrangian $\mathcal{L}$ on $\mathcal{U}$ is a mapping

$$\mathcal{L} : C^\infty_c(M) \to \mathcal{F}_c(B, \mathcal{U}),$$

such that

GL1. $\mathrm{supp}(\mathcal{L}(f)) \subseteq \mathrm{supp}(f)$ and $\mathcal{L}(f)$ is B-smooth for all $f \in C^\infty_c(M)$,
GL2. for each f_1, f_2, $f_3 \in C^\infty_c(M)$ with $\mathrm{supp}(f_1) \cap \mathrm{supp}(f_3) = \emptyset$,

$$\mathcal{L}(f_1 + f_2 + f_3) = \mathcal{L}(f_1 + f_2) - \mathcal{L}(f_2) + \mathcal{L}(f_2 + f_3).$$

Given the properties of the above Definition we immediately get:

Proposition 3.12. *Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open, $\mathcal{L}$ a generalized Lagrangian on $\mathcal{U}$. Then*

i) $\mathrm{supp}(\mathcal{L}(f + f_0) - \mathcal{L}(f_0)) \subseteq \mathrm{supp}(f)$ *for all f, $f_0 \in C^\infty_c(M)$,*
ii) for all $f \in C^\infty_c(M)$, $\mathcal{L}(f)$ is a local functional.

The proof of this result can be found in Lemma 3.1 and Lemma 3.2 in [BFR19].

Combining the linearity of $C^\infty_c(M)$, property GL2 and Proposition 3.12 we obtain that each generalized Lagrangian can be written as a suitable sum of arbitrarily small supported generalized Lagrangians. To see it, fix $\epsilon > 0$ and consider $\mathcal{L}(f)$. By compactness $\mathrm{supp}(f)$ admits a finite open cover of balls, $\{B_i\}_{i \in I}$ of radius ϵ such that none of the open balls is completely contained in the union of the others. Let $\{g_i\}_{i \in I}$ be a partition of unity subordinate to the above cover of $\mathrm{supp}(f)$, set $f_i \doteq g_i \cdot f$. Then using GL2

$$\mathcal{L}(f) = \mathcal{L}\left(\sum_i f_i\right) = \sum_{J \subset I} c_J \mathcal{L}\left(\sum_{j \in J} f_j\right),$$

where $J \subset I$ contains the indices of all balls B_i having non empty intersection with a fixed ball (the latter included), and $c_J = \pm 1$ are suitable coefficients determined by the application of GL2. By construction each index J has at most two elements and $\mathrm{supp}(\sum_{j\in J} f_j)$ is contained at most in a ball of radius 2ϵ. We have thus split $\mathcal{L}(f)$ as a sum of generalized Lagrangians with arbitrarily small supports.

Definition 3.13. Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open, $\mathcal{L}$ a generalized Lagrangian on $\mathcal{U}$. The k-th Euler-Lagrange derivative of $\mathcal{L}$ in $\varphi \in \mathcal{U}$ along $(\vec{X}_1, \dots, \vec{X}_k) \in \Gamma^\infty_c(M \leftarrow \varphi^* VB)^k$ is

$$\delta^{(k)}\mathcal{L}(1)_\varphi[0](\vec{X}_1, \dots, \vec{X}_k) \doteq \left.\frac{d^k}{dt_1 \dots dt_k}\right|_{t_1=\dots=t_k=0} \mathcal{L}(f)_\varphi[0](t_1\vec{X}_1 + \dots + t_k\vec{X}_k) \tag{25}$$

where $f|_K \equiv 1$ on a suitable compact K containing all compacts $\mathrm{supp}(\vec{X}_i)$.

One can see how the compact supports of the Γ^∞-tangent vectors, i.e. the sections of $\Gamma^\infty_c(M \leftarrow \varphi^*VB)^k$ allow us to perform an adiabatic limit and consider the cutoff function f to be identically 1 throughout M.

From now on we will assume that generalized Lagrangian used are microlocal, i.e. $\mathcal{L}(f) \in \mathcal{F}_{\mu loc}(B, \mathcal{U})$ for each $f \in C^\infty_c(M)$; this means that the first Euler-Lagrange derivative can be written as

$$\delta^{(1)}\mathcal{L}(1)_\varphi[0](\vec{X}) = \int_M E(\mathcal{L})_\varphi[0](\vec{X}), \tag{26}$$

where by microlocality $E(\mathcal{L})_\varphi[0] \in \Gamma^\infty_c(M \leftarrow \varphi^*VB' \otimes \Lambda_m(M))$.

A generalized Lagrangian $\mathcal{L}$ is trivial whenever $\mathrm{supp}\big(\mathcal{L}(f)\big) \subset \mathrm{supp}(df)$ for each $f \in C^\infty_c(M)$. Triviality induces an equivalence relation on the space of generalized Lagrangians, namely two $\mathcal{L}_1$, $\mathcal{L}_2$ are equivalent whenever their difference is trivial. We can show that if two Lagrangians $\mathcal{L}_1$, $\mathcal{L}_2$ are equivalent then they end up producing the same first variation (26). For instance suppose that $\mathcal{L}_1(f) - \mathcal{L}_2(f) = \Delta\mathcal{L}(f)$ with $\Delta\mathcal{L}(f)$ trivial generalized Lagrangian for each $f \in C^\infty_c(M)$. To evaluate $\delta^{(1)}\Delta\mathcal{L}(1)_\varphi[0](\vec{X})$ one has to choose some f which is identically 1 in a neighborhood of $\mathrm{supp}(\vec{X})$, however, by **GL1** in Definition 3.11 $\mathrm{supp}\big(\Delta\mathcal{L}(f)\big) \subset \mathrm{supp}(df) \cap \mathrm{supp}(\vec{X}) = \emptyset$, therefore by Lemma 3.4 we obtain $E(\Delta\mathcal{L})_\varphi[0](\vec{X}) = 0$ and

$$\begin{aligned}\delta^{(1)}\mathcal{L}_1(f)_\varphi[0](\vec{X}) &= \delta^{(1)}\mathcal{L}_2(1)_\varphi[0](\vec{X}) + \delta^{(1)}\Delta\mathcal{L}(1)_\varphi[0](\vec{X}) \\ &= \int_M E(\mathcal{L}_2)_\varphi[0](\vec{X}) + \int_M E(\Delta\mathcal{L})_\varphi[0](\vec{X}) \\ &= \delta^{(1)}\mathcal{L}_2(1)_\varphi[0](\vec{X}).\end{aligned}$$

Finally we compare our generalized action functional with the *standard* action which is generally used in classical field theory (see e.g. [FF03], [Kru15]). One generally introduce the *standard geometric* Lagrangian, λ of order r, as a bundle morphism

$$\begin{tikzcd} J^rB \arrow[r, "\lambda"] \arrow[d, "\pi^r"] & \Lambda_m(M) \arrow[d, "\rho"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

between (J^rB, π^r, M) and $(\Lambda_m(M), \rho, M, |\wedge^m T^*_xM|)$, where the latter is the vector bundle whose typical fiber is the vector space of weight one m-form densities. In coordinates, setting $d\sigma(x) = dx^1 \wedge \dots \wedge dx^m$ we can write $\lambda(j^r_xy) = \lambda(j^r_xy)d\sigma(x)$. Two Lagrangian morphisms λ_1, λ_2 are equivalent whenever their difference is an exact form. Its associated *standard geometric* action functional will therefore be

$$\mathcal{A}_D(\varphi) = \int_M \chi_D(x)\lambda(j^r_x\varphi)d\sigma(x) \tag{27}$$

where λ an element of the equivalence class of Lagrangian morphisms, D is a compact region of M whose boundary ∂D is an orientable $(m-1)$-manifold and χ_D its characteristic function. One could be tempted to draw a parallel with a generalized Lagrangian by considering the mapping

$$\chi_D \mapsto \mathcal{A}(\chi_D) = \int_M \chi_D(x)\lambda(j^r_x\varphi)d\sigma(x)\ . \tag{28}$$

However (28) differs from Definition 3.11 in the singular character of the cutoff function. Indeed the functional $\mathcal{A}_D \in \mathcal{F}_{loc}(B, \mathcal{U})$ for each choice of compact D but it is *never* microlocal, for the integral kernel of $\mathcal{A}(\chi_D)^{(1)}_\varphi[0]$ has always singularities localized in ∂D. This is a severe problem when attempting to calculate the Peierls bracket for local functionals, indeed we can extend this bracket to less regular functionals (see Definition 4.7 and Theorem 4.11) mantaining the closure of the operation (see Theorem 4.11); however, we cannot outright extend the bracket to all local functionals. Therefore, in order to accommodate those less regular functionals such as (27), one would need to place severe restrictions on the possible compact subsets D which cut off possible integration divergences. This, however, is not consistent with the derivation of Euler-Lagrange equations by the usual variation technique where the latter are obtained by imposing requirements that ought to hold *for each* $D \subset M$ compact.

Of course, given a Lagrangian morphism λ of order r we can always define a generalized microlocal Lagrangian by a microlocal-valued distribution, *i.e.*

$$C_c^\infty(M) \times \mathcal{U} \ni (f,\varphi) \mapsto \mathcal{L}(f)(\varphi) = \int_M f(x)\lambda(j_x^r\varphi)d\sigma(x) \ .$$

When we calculate higher order derivatives we get

$$d^{(k+1)}\mathcal{L}(1)_\varphi[0](\vec{X}_1,\ldots,\vec{X}_{k+1}) = \int_M \delta^{(k)}E(\mathcal{L})_\varphi[0](\vec{X}_1)(\vec{X}_2,\ldots,\vec{X}_{k+1}). \tag{29}$$

In particular, we can view $\delta^{(1)}E(\mathcal{L})_\varphi[0] : \Gamma_c^\infty(M \leftarrow \varphi^*VB) \to \Gamma_c^\infty\big(M \leftarrow \varphi^*VB' \otimes \Lambda_m(M)\big)$, and induce the linearized field equations around φ which represents one of the ingredients for the construction of the Peierls bracket.

4. The Peierls bracket

Heuristically speaking the Peierls bracket is a duality relating two obsevables, F, G, that accounts for the effect of the (antisymmetric) influence of F on G when the latter is perturbed around a solution of certain equations. We will define this quantity using the linearized field equations which can be constructed with the second derivative of a generalized Lagrangian, which with some additional hypotheses will turn out to be normally hyperbolic. We start by reviewing some basic notions from the theory of normally hyperbolic (NH) operators.

A linear map $D : \Gamma^\infty(M \leftarrow E) \to \Gamma^\infty(M \leftarrow E)$ such that

(*i*) D can be restricted to a linear map $D_U : \Gamma^\infty(U \leftarrow E\mid_U) \to \Gamma^\infty(U \leftarrow E\mid_U)$ on open subset U of M such that $\forall \vec{s} \in \Gamma^\infty(M \leftarrow E)$

$$D_U\left(\vec{s}|_U\right) = (D\vec{s})\mid_U \ ,$$

(*ii*) in a local chart $(U, \{x^\mu\})$ with local base sections $\{e_j\} \in \Gamma^\infty(U \leftarrow E\mid_U)$, there are smooth coefficients $D_j^{\mu_1,\ldots,\mu_r,i}$ totally symmetric in the Greek indices, for which

$$D\vec{s}\mid_U = \sum_{0\le r\le k} \frac{1}{r!} D_j^{i,\ \mu_1,\ldots,\mu_r} \frac{\partial^r s^j}{\partial x^{\mu_1}\ldots\partial x^{\mu_r}} e_i,$$

we call it a linear differential operator of order k. The principal symbol of the differential operator D is the element

$$\sigma_k(D)\mid_U = D_j^{\mu_1,\ldots,\mu_k,i} \frac{\partial}{\partial x^{\mu_1}} \otimes\ldots\otimes \frac{\partial}{\partial x^{\mu_k}} \otimes e^j \otimes e_i \in \Gamma^\infty(M \leftarrow \vee^k TM) \otimes \mathrm{End}(E).$$

Definition 4.1. Let (E,π,M,V) be a vector bundle with base a Lorentzian manifold, a second order differential operator $D : \Gamma^\infty(M \leftarrow E) \to \Gamma^\infty(M \leftarrow E)$ is called normally hyperbolic if its principal symbol can be written as

$$\sigma_2(D) = g^{-1} \otimes \mathrm{id}_E$$

where g is the Lorentzian metric on M.

In the last section we have shown how for a microlocal generalized Lagrangian $\delta^{(1)}E(\mathcal{L})_\varphi[0] : \Gamma_c^\infty(M \leftarrow \varphi^*VB) \otimes \Gamma_c^\infty(M \leftarrow \varphi^*VB) \to \Lambda_m(M)$ one can define a linear operator

$$\delta^{(1)}E(\mathcal{L})_\varphi[0] : \Gamma_c^\infty(M \leftarrow \varphi^*VB) \to \Gamma_c^\infty\big(M \leftarrow \varphi^*VB' \otimes \Lambda_m(M)\big)$$

if we fix a metric h on the standard fiber of B and the Lorentzian metric g of M inducing the Hodge isomorfism $*_g$, we can define

$$D_\varphi \doteq (\varphi^*h)^\sharp \circ (\mathrm{id}_{\varphi^*VB'} \otimes *_g) \circ \delta^{(1)}E(\mathcal{L})_\varphi[0] : \Gamma^\infty(M \leftarrow \varphi^*VB) \to \Gamma^\infty(M \leftarrow \varphi^*VB). \tag{30}$$

For each φ we can see that D_φ is a differential operator and determine the principal symbol. Note in particular that once the principal symbol is known with respect to some section, say φ, then it is known with respect to any other section in $\mathcal{U}_\varphi$, by (12)

$$\delta^{(1)}E(\mathcal{L})_\psi[0](\vec{X}_1,\vec{X}_2) = \delta^{(1)}E(\mathcal{L})_\varphi[0]\left(d^{(1)}u_{\varphi\psi}[u_\psi(\varphi)](\vec{X}_1), d^{(1)}u_{\varphi\psi}[u_\psi(\varphi)](\vec{X}_2)\right) + E(\mathcal{L})_\varphi[0]\left(d^{(2)}u_{\varphi\psi}[u_\psi(\varphi)](\vec{X}_1,\vec{X}_2)\right);$$

while the second piece modifies the expression of the differential operator, it does not alter its principal symbol since the local form of $d^{(2)}u_{\varphi\psi}[u_\psi(\varphi)](\vec{X}_1,\vec{X}_2)$ does yield extra derivatives. We therefore arrive at the conclusion that if we use a generalized Lagrangian $\mathcal{L}$ whose linearized equations differential operator, D_φ, is normally hyperbolic for some $\varphi_0\in\mathcal{U}$, then it is normally hyperbolic (with the same principal symbol) for all $\varphi\in\mathcal{U}_{\varphi_0}$.

Let us give an example of a microlocal generalized Lagrangian that has normally hyperbolic linearized equations. Recalling formula (16) with $\lambda : J^1(M\times N)\to\Lambda^m(M)$, the latter being a first order Lagrangian, we have

$$\mathcal{L}^{(2)}_{f,\lambda,\varphi}[0](\vec{X}_1,\vec{X}_2) = \int_M f(x)\bigg\{\frac{\partial^2\lambda}{\partial y^i\partial y^j}\vec{X}_1^i\vec{X}_2^j + \frac{\partial^2\lambda}{\partial y^i_\mu\partial y^j}d_\mu\big(\vec{X}_1^i\big)\vec{X}_2^j + \frac{\partial^2\lambda}{\partial y^i_\mu\partial y^j}\vec{X}_1^i d_\mu\big(\vec{X}_2^j\big) + \frac{\partial^2\lambda}{\partial y^i_\mu\partial y^j_\nu}d_\mu\big(\vec{X}_1^i\big)d_\nu\big(\vec{X}_2^j\big)\bigg\}(x)d\mu_g(x).$$

The key ingredient for the principal symbol is the quantity $m^{\mu\nu}_{ij}\doteq\frac{\partial^2\lambda}{\partial y^i_\mu\partial y^j_\nu}$. Applying the transformations to get the differential operator of linearized field equations, as in (30), to the above quantity yields principal symbol

$$\sigma_2(D_\varphi) = h^{ij}m^{\mu\nu}_{jk}\otimes\partial_\mu\vee\partial_\nu\otimes e_i\otimes e^j. \tag{31}$$

In case this quantity satisfies the condition of Definition 4.1 we can conclude that the operator is normally hyperbolic. There are also other notions of hyperbolicity, for instance see [Chr00], where his hyperbolicity condition is strictly weaker then the one employed here. From now on we shall assume that our microlocal Lagrangian produces always normally hyperbolic linearized equations. Then referring to the general case we have the following theorem:

Theorem 4.2. *Let (E,π,M,V) be a vector bundle with base a globally hyperbolic Lorentzian manifold and let $D\in\mathrm{DiffOp}^2(E)$ be a normally hyperbolic differential operator. Then D admits global Green operators $G^\pm_M:\Gamma^\infty_c(M\leftarrow E)\to\Gamma^\infty(M\leftarrow E)$ and their causal propagator $G_M:\Gamma^\infty_c(M\leftarrow E)\to\Gamma^\infty(M\leftarrow E)$, satisfying the following properties:*

- (i) *Continuity. $G^\pm_M$, G_M are continuous[6] linear operators admitting a continuous and linear extension to the space $\Gamma^{-\infty}_\pm(M\leftarrow E)$ topological dual to the space $\Gamma^\infty_\mp(M\leftarrow E)=\{\vec{u}\in\Gamma^\infty(M\leftarrow E):\ \forall p\in M\ \mathrm{supp}(\vec{u})\cap J^\mp_M(p)\text{ is compact}\}$.*
- (ii) *Support Properties.*

$$\mathrm{supp}(G^\pm_M\vec{u})\subset J^\pm_M(\mathrm{supp}(\vec{u}))$$

for all $\vec{u}\in\Gamma^{-\infty}_\pm(E)$.
- (iii) *Wave Front Sets.*

$$\begin{aligned}\mathrm{WF}(G^\pm_M) &= \{(x,x,\xi,-\xi)\in T^*(M\times M)\text{ with }(x,\xi)\in T^*M\backslash 0\}\\ &\quad\cup\{(x_1,x_2,\xi_1,\xi_2)\in T^*(M\times M)\text{ with }(x_1,x_2,\xi_1,-\xi_2)\in\mathrm{BiCh}^g_\gamma\},\\ \mathrm{WF}(G_M) &= \{(x_1,x_2,\xi_1,\xi_2)\in T^*(M\times M)\text{ with }(x_1,x_2,\xi_1,-\xi_2)\in\mathrm{BiCh}^g_\gamma\};\end{aligned}$$

where BiCh^g_γ is the bicharacteristic strip of a lightlike geodesic γ, i.e. the set of points (x_1,x_2,ξ_1,ξ_2) such that there is an interval $[0,\Lambda]\subset\mathbb{R}$ for which $(x_1,\xi_1)=(\gamma(0),g^\flat\dot\gamma(0))$ and $(x_2,-\xi_2)=(\gamma(\Lambda),g^\flat\dot\gamma(\Lambda))$.
- (iv) *Propagation of Singularities. Given $\vec{u}\in\Gamma^{-\infty}_\pm(E)$ we have that $(x,\xi)\in\mathrm{WF}(G^\pm_M(\vec{u}))$ if either $(x,\xi)\in\mathrm{WF}(\vec{u})$ or there is a lightlike geodesic γ and some $(y,\eta)\in\mathrm{WF}(\vec{u})$ such that $(x,y,\xi,-\eta)\in\mathrm{BiCh}^g_\gamma$. Similarly $(x,\xi)\in\mathrm{WF}(G_M(\vec{u}))$ if there is a lightlike geodesic γ and some $(y,\eta)\in\mathrm{WF}(\vec{u})$ such that $(x,y,\xi,-\eta)\in\mathrm{BiCh}^g_\gamma$.*

In the above theorem the notation $\Gamma^{-\infty}(M\leftarrow E)$ denotes distributional sections of the vector bundle E, i.e. continuous linear mappings : $\Gamma^\infty_c(M\leftarrow E)\to\mathbb{C}$, where the first space is endowed with the usual limit Fréchet topology. We also recall that the wave front set at $x\in M$ of a distributional section $\vec{u}$ of a vector bundle

[6]In the Fréchet locally convex topology of $\Gamma^\infty(M\leftarrow E)$ and the limit Fréchet topology of $\Gamma^\infty_c(M\leftarrow E)$.

E of rank k is calculated as follows: fixing a trivialization (U_α, t_α) on E, then locally $\vec{u}$ is represented by k distributions $u^i \in \mathcal{D}'(U_\alpha)$, each of which will have its own wave front set. Then we set

$$\mathrm{WF}(\vec{u}) \doteq \bigcup_{i=1}^{k} \mathrm{WF}(u^i). \tag{32}$$

It is possible to show that choosing a different trivialization in the vector bundle give rise to a smooth vertical fibered morphism which does not alter the wave front set, as a result (32) is independent of the trivialization chosen and hence well defined.

Summing up we created a way of associating to each φ in the domain of $\mathcal{L}$, an operator

$$G^{\pm}_{\varphi} : \Gamma^\infty_c(M \leftarrow \varphi^* VB) \to \Gamma^\infty(M \leftarrow \varphi^* VB). \tag{33}$$

By (i) of Theorem 4.2 and linearity, $G^{\pm}_{\varphi}$ is a smooth mapping, that is, $G^{\pm}_{\varphi} \in C^\infty\big(\Gamma^\infty_c(M \leftarrow \varphi^* VB), \Gamma^\infty(M \leftarrow \varphi^* VB)\big)$ for each $\varphi \in \mathcal{U}$. Given $\vec{s} \in \Gamma^{-\infty}(M \leftarrow \varphi^* VB)$ we can view

$$G^{\pm}(\vec{s}) : \mathcal{U} \ni \varphi \mapsto G^{\pm}_{\varphi}(\vec{s}) \in \Gamma^\infty(M \leftarrow \varphi^* VB).$$

We ask whether this map is MB-smooth, in particular we seek to evaluate

$$\lim_{t\to 0} \frac{G^{\pm}_{u_\varphi^{-1}(t\vec{X})}(\vec{s}) - G^{\pm}_{\varphi}(\vec{s})}{t}. \tag{34}$$

Lemma 4.3. *Let $\gamma : \mathbb{R} \to \mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be a smooth curve, then for each fixed $\vec{X} \in \Gamma^\infty_c(\varphi^* VB)$ the mapping $t \mapsto G^{\pm}_{\gamma(t)}(\vec{X}) \in \Gamma^\infty(M \leftarrow \varphi^* VB)$ is smooth. In particular we have*

$$G^{\pm\ (1)}_{\varphi}(\vec{X}) = \lim_{t\to 0} \frac{1}{t}\left(G^{\pm}_{u_\varphi^{-1}(t\vec{X})} - G^{\pm}_{\varphi}\right) = -G^{\pm}_{\varphi} \circ D^{(1)}_{\varphi}(\vec{X}) \circ G^{\pm}_{\varphi}, \tag{35}$$

where $\mathcal{U} \ni \varphi \mapsto D_\varphi(\vec{X}) \in \Gamma^\infty(M \leftarrow \varphi^* VB)$ is the mapping induced by (30).

Proof. We just show the claim for the retarded propagator since for the advanced one the result follows in complete analogy. We start by evaluating

$$\lim_{t\to 0} \frac{1}{t}\Big(G^{+}_{\gamma(t)}(\vec{s}) - \mathcal{G}^{+}_{\gamma(0)}(\vec{s})\Big)(\vec{Y})$$

In the following argument we will omit the evaluation at $\vec{s}$ from the notation. The differential operator $D(\vec{Y}) : \mathcal{U} \ni \varphi \mapsto D_\varphi(\vec{Y}) \in \Gamma^\infty(M \leftarrow \varphi^* VB)$ is smooth when composed with γ, therefore we consider

$$\begin{aligned}
D_{\gamma(t)} \lim_{t\to 0} \frac{1}{t}\Big(G^{+}_{\gamma(t)} - G^{+}_{\gamma(0)}\Big)(\vec{Y}) &= \lim_{t\to 0} \frac{1}{t}\Big(D_{\gamma(t)} G^{+}_{\gamma(t)} - D_{\gamma(t)} G^{+}_{\gamma(0)}\Big)(\vec{Y}) \\
&= \lim_{t\to 0} \frac{1}{t}\Big(\mathrm{id}_{\Gamma^\infty(M\leftarrow\gamma(0)^* VB)} - D_{\gamma(t)} G^{+}_{\gamma(0)}\Big)(\vec{Y}) \\
&= \lim_{t\to 0} \frac{1}{t}\Big(D_{\gamma(0)} G^{+}_{\gamma(0)} - D_{\gamma(t)} G^{+}_{\gamma(0)}\Big)(\vec{Y}) \\
&= \lim_{t\to 0} \frac{1}{t}\Big(D_{\gamma(0)} - D_{\gamma(t)}\Big)\big(G^{+}_{\gamma(0)}(\vec{Y})\big) \\
&= -D^{(1)}_{\gamma(0)}(\dot{\gamma}(0)) \circ G^{+}_{\gamma(0)}(\vec{Y}).
\end{aligned}$$

since γ is a smooth curve in $\Gamma^\infty(M \leftarrow B)$, given any interval $[-\epsilon, \epsilon]$ with $\epsilon > 0$, there is a compact subset of M for which $\gamma(t)(x)$ is constant in t on $M \backslash K_\epsilon$, then differential operator $D^{(1)}_{\gamma(0)} \neq 0$ only inside K_ϵ, therefore the quantity $\big(D^{(1)}_{\gamma(0)}(\dot{\gamma}(0)) \circ G^{\pm}_{\gamma(0)}\big)(\vec{Y})$ has compact support for any $\vec{Y} \in \Gamma^\infty(M \leftarrow \gamma(0)^* VB)$ and we can write

$$\lim_{t\to 0} \frac{1}{t}\Big(G^{+}_{\gamma(t)} - \mathcal{G}^{+}_{\gamma(0)}\Big) = -G^{+}_{\gamma(0)} \circ D^{(1)}_{\gamma(0)}(\dot{\gamma}(0)) \circ G^{+}_{\gamma(0)}.$$

Using the above relation one can show that all iterated derivatives of G^{+}_{φ} exists, thus showing smoothness. □

Similarly for the causal propagator we find

$$G^{(1)}_{\varphi}(\vec{X}) \doteq \lim_{t\to 0} \frac{1}{t}\left(G_{u_\varphi^{-1}(t\vec{X})} - G_{\varphi}\right) = -G_{\varphi} \circ D^{(1)}_{\varphi}(\vec{X}) \circ G^{+}_{\varphi} - G^{-}_{\varphi} \circ D^{(1)}_{\varphi}(\vec{X}) \circ G_{\varphi}. \tag{36}$$

Given the Green's functions $G^{\pm}_{\varphi}$, set

$$\mathcal{G}^{\pm}_{\varphi} \doteq G^{\pm}_{\varphi} \circ (\varphi^* h)^\sharp \circ (id_{(\varphi^* VB)'} \otimes *_g) : \Gamma^\infty_c(M \leftarrow (\varphi^* VB)' \times \Lambda_m(M)) \to \Gamma^\infty(M \leftarrow \varphi^* VB), \tag{37}$$

$$\mathcal{G}_{\varphi} \doteq G_{\varphi} \circ (\varphi^* h)^\sharp \circ (id_{(\varphi^* VB)'} \otimes *_g) : \Gamma^\infty_c(M \leftarrow (\varphi^* VB)' \times \Lambda_m(M)) \to \Gamma^\infty(M \leftarrow \varphi^* VB). \tag{38}$$

Note how, up to this point, we used some fiberwise metric h in (30) in order to have a proper differential operator for the subsequent steps. As a consequence the resulting operator $D_\varphi(h)$ does depend on the metric chosen and so do its retarded and advanced Green's operators $G^\pm_\varphi(h)$ with their counterparts $\mathcal{G}^\pm_\varphi(h)$. From the definition of Green's operators we have

$$\begin{cases} D_\varphi(h)\circ G^\pm_\varphi(h) = \mathrm{id}_{\Gamma^\infty_c(M\leftarrow\varphi^*VB)}\ , \\ G^\pm_\varphi(h)\circ D_\varphi(h)\big|_{\Gamma^\infty_c(M\leftarrow\varphi^*VB)} = \mathrm{id}_{\Gamma^\infty_c(M\leftarrow\varphi^*VB)}\ . \end{cases}$$

The latter is equivalent to

$$\begin{cases} \delta^{(1)}E(\mathcal{L})_\varphi[0]\circ \mathcal{G}^\pm_\varphi(h) = \mathrm{id}_{\Gamma^\infty_c(M\leftarrow\varphi^*VB'\otimes\Lambda_m(M))}\ , \\ \mathcal{G}^\pm_\varphi(h)\circ \delta^{(1)}E(\mathcal{L})_\varphi)[0]\big|_{\Gamma^\infty_c(M\leftarrow\varphi^*VB)} = \mathrm{id}_{\Gamma^\infty_c(M\leftarrow\varphi^*VB)}\ . \end{cases}$$

Using the notation of [Bär15], the family of operators $\{\mathcal{G}^\pm_\varphi(h)\}$ defines Green-hyperbolic type of operators with respect to the differential operator of the linearized equations at φ. Finally, using Theorem 3.8 in [Bär15], we get uniqueness for the advanced and retarded propagators, which in turn results in the independence on the Riemannian metric h used before. The idea behind the proof is as follows: one would like to both extend the domain of $\mathcal{G}^\pm_\varphi(h)$ and reduce the target space to the same suitable space, once this is done, each propagator becomes the inverse of the linearized equations, then using uniqueness of the inverse we conclude. It turnes out that the extension to the spaces $\Gamma^\infty_{\mp}(M\leftarrow\varphi^*VB'\otimes\Lambda_m(M))$ of future/past compact smooth sections does the job.

Lemma 4.4. *Let g a Lorentzian metric on M and $D:\Gamma^\infty(M\leftarrow E)\to\Gamma^\infty(M\leftarrow E)$ a linear partial differential operator. Then D is self adjoint with respect to the pairing[7] given by*

$$\langle \vec{s},\vec{t}\,\rangle = \int_M (id_{E'}\otimes *_g\circ h^\flat(\vec{t}\,))\vec{s} = \int_M h^\flat(\vec{t}\,)\vec{s}\ d\mu_g$$

if and only if its kernel $D(x,y)$ is symmetric. Moreover if D is normally hyperbolic, then G^+_M and G^-_M are each the adjoint of the other in the common domain.

Proof. The equivalent condition follows essentially from following chain of equivalences

$$\begin{aligned}\langle D\vec{s},\vec{t}\,\rangle &= \int_M h^\flat(\vec{t})(x)D\vec{s}(x)\ d\mu_g(x) = \int_M h^\flat(\vec{t})(x)h^\sharp\circ(id_{E'}\otimes *_g)\mathcal{D}(x,\vec{s})\ d\mu_g(x)\\ &= \int_{M^2}\mathcal{D}(x,y)\vec{t}(x)\vec{s}(y)d\mu_g(x)d\mu_g(y)\end{aligned}\ .$$

Suppose now D is self adjoint, then

$$\langle \vec{s},G^-_M\vec{t}\,\rangle = \langle DG^+_M\vec{s},G^-_M\vec{t}\,\rangle = \langle G^+_M\vec{s},DG^-_M\vec{t}\,\rangle = \langle G^+_M\vec{s},\vec{t}\,\rangle$$

whence the desired adjoint properties of G^+_M and G^-_M. □

For future convenience, we calculate the functional derivatives of $\mathcal{G}^\pm_\varphi$ and $\mathcal{G}_\varphi$, which are clearly smooth by combining Lemma 4.3 with (37) and (38), whence

$$d^k\mathcal{G}^\pm_\varphi(\vec{X}_1,\dots,\vec{X}_k) = \sum_{l=1}^k(-1)^l \sum_{\substack{(I_1,\dots,I_l)\\ \in\mathcal{P}(1,\dots,k)}}\left(\prod_{i=1}^l \mathcal{G}^\pm_\varphi\circ\delta^{(|I_{\sigma(i)}|+1)}E(L)_\varphi[0]\big(\vec{X}_{I_i}\big)\right)\circ\mathcal{G}^\pm_\varphi, \tag{39}$$

$$\begin{aligned}&d^k\mathcal{G}_\varphi(\vec{X}_1,\dots,\vec{X}_k)\\ &\quad=\sum_{l=1}^k(-1)^l\sum_{(I_1,\dots,I_l)\in\mathcal{P}(1,\dots,k)}\sum_{m=0}^l\left(\prod_{i=1}^m\mathcal{G}^-_\varphi\circ\delta^{(|I_i|+1)}E(L)\big(\vec{X}_{I_i}\big)\right)\circ\mathcal{G}_\varphi\circ\left(\prod_{i=m+1}^l\delta^{(|I_i|+1)}E(L)_\varphi\big(\vec{X}_{I_i}\big)\circ\mathcal{G}^+_\varphi\right),\end{aligned} \tag{40}$$

where $(I_1,\dots,I_l)$ is partition of the set $\{1,\dots,k\}$, and $\vec{X}_I=\otimes_{i\in I}\vec{X}_i$. The main takeaway from (40) is the pattern of the composition of propagators and derivatives of $E(L)$, that is first the $\mathcal{G}^-_\varphi$'s, then a single $\mathcal{G}$ and at the end some $\mathcal{G}^+_\varphi$'s intertwined by derivatives of $E(\mathcal{L})$. These will be key to some later proofs. We are now in a position to introduce the Peierls bracket:

[7]We require that at least one of the entries has compact support.

Definition 4.5. Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open, and $F,\ G \in \mathcal{F}_{\mu loc}(B,\mathcal{U})$. Fix a generalized microlocal Lagrangian $\mathcal{L}$ whose linearized equations induce a normally hyperbolic operator. The retarded and advanced products $\mathsf{R}_\mathcal{L}(F,G)$, $\mathsf{A}_\mathcal{L}(F,G)$ are functionals defined by

$$\mathsf{R}_\mathcal{L}(F,G)(\varphi) \doteq \left\langle F^{(1)}_\varphi[0], \mathcal{G}^+_\varphi G^{(1)}_\varphi[0] \right\rangle, \tag{41}$$

$$\mathsf{A}_\mathcal{L}(F,G)(\varphi) \doteq \left\langle F^{(1)}_\varphi[0], \mathcal{G}^-_\varphi G^{(1)}_\varphi[0] \right\rangle, \tag{42}$$

while the Peierls bracket of F and G is

$$\{F,G\}_\mathcal{L} \doteq \mathsf{R}_\mathcal{L}(F,G) - \mathsf{A}_\mathcal{L}(F,G). \tag{43}$$

We recall that for a microlocal functional F, by (15)

$$F^{(1)}_\varphi[0](\vec{X}) = \int_M f^{(1)}_\varphi[0]_i(x) X^i(x) d\mu_g(x),$$

therefore we can write $\{F,G\}_\mathcal{L}(\varphi)$ as

$$\int_{M^2} f^{(1)}_\varphi[0]_i(x) \mathcal{G}^{ij}_\varphi(x,y) g^{(1)}_\varphi[0]_j(y) d\mu_g(x,y) \tag{44}$$

where repeated indices as usual follows the Einstein notation. This implies clearly that Definition 4.5 is well posed. Moreover as a consequence of Lemma 4.4 we see that the Peierls bracket of F and G can also equivalently viewed as $\mathsf{R}_\mathcal{L}(F,G) - \mathsf{R}_\mathcal{L}(G,F) = \mathsf{A}_\mathcal{L}(G,F) - \mathsf{A}_\mathcal{L}(F,G)$.

We begin our analysis of the Peierls bracket by listing the support properties of the functionals defined in Definition 4.5.

Proposition 4.6. *Let $\mathcal{U}$, F, G be as in the above definition, then the retarded, advanced products and Peierls bracket are B-smooth with the following support properties:*

$$\operatorname{supp}\left(R_\mathcal{L}(F,G)\right) \subset J^+(\operatorname{supp}(F)) \cap J^-(\operatorname{supp}(G)), \tag{45}$$

$$\operatorname{supp}\left(A_\mathcal{L}(F,G)\right) \subset J^+(\operatorname{supp}(G)) \cap J^-(\operatorname{supp}(F)), \tag{46}$$

which combined yields

$$\operatorname{supp}\left(\{F,G\}_\mathcal{L}\right) \subset \left(J^+(\operatorname{supp}(F)) \cup J^-(\operatorname{supp}(F))\right) \cap \left(J^+(\operatorname{supp}(G)) \cup J^-(\operatorname{supp}(G))\right). \tag{47}$$

Proof. By definition the support properties of $\mathcal{G}^\pm_M$ and $G^\pm_M$ are analogue, so combining these properties with $\mathsf{R}_\mathcal{L}(F,G) = \frac{1}{2}\mathsf{R}_\mathcal{L}(F,G) + \frac{1}{2}\mathsf{A}_\mathcal{L}(G,F)$ yields the desired result. We now turn to the smoothness. We calculate the k-th derivative of $\mathsf{R}_\mathcal{L}$. By the chain rule, taking $\mathcal{P}(1,\ldots,k)$ the set of permutations of $\{1,\ldots,k\}$, we can write

$$\begin{aligned} &\mathsf{R}_\mathcal{L}(F,G)^{(k)}_\varphi[0](\vec{X}_1,\ldots,\vec{X}_k) \\ &\quad= \sum_{(J_1,J_2,J_3)\subset P_k} \left\langle F^{(|J_1|+1)}_\varphi[0](\otimes_{j_1\in J_1}\vec{X}_{j_1})\ ,\ d^{(|J_2|)}\mathcal{G}^+_\varphi(\otimes_{j_2\in J_2}\vec{X}_{j_2}) G^{(|J_3|+1)}_\varphi[0](\otimes_{j_3\in J_3}\vec{X}_{j_3}) \right\rangle, \end{aligned} \tag{48}$$

and similarly

$$\begin{aligned} &\mathsf{A}_\mathcal{L}(F,G)^{(k)}_\varphi(\vec{X}_1,\ldots,\vec{X}_k) \\ &\quad= \sum_{(J_1,J_2,J_3)\subset P_k} \left\langle F^{(|J_1|+1)}[\varphi](\otimes_{j_1\in J_1}\vec{X}_{j_1})\ ,\ d^{(|J_2|)}\mathcal{G}^-_\varphi(\otimes_{j_2\in J_2}\vec{X}_{j_2}) G^{(|J_3|+1)}_\varphi[0](\otimes_{j_3\in J_3}\vec{X}_{j_3}) \right\rangle. \end{aligned} \tag{49}$$

To see that the pairing in the derivatives of the advanced, retarded products are well defined, we use the kernel notation (15), therefore we write the integral kernel of $\mathsf{R}_\mathcal{L}(F,G)$, which by a little abuse of notation we call $\mathsf{R}_\mathcal{L}(F,G)(x,y)$ for x,$y \in M$. It is

$$\mathsf{R}_\mathcal{L}(F,G)(x,y) = f^{(1)}_\varphi[0]_i(x) \left(\mathcal{G}^+_\varphi\right)^{ij}(x,y) g^{(1)}_\varphi[0]_j(y).$$

Using this notation, we can write the integral kernel $\mathsf{R}_\mathcal{L}(F,G)^{(k)}_\varphi[0](\vec{X}_1,\ldots,\vec{X}_k)(x,y)$ in (48) as a sum of terms with two possible contributions:

case $J_2 = \emptyset$:

$$f_\varphi^{(p+1)}[0]_i(x, \vec{X}_1, \ldots, \vec{X}_p)(\mathcal{G}_\varphi^+)^{ij}(x,y) g_\varphi^{(q+1)}[0]_j(y, \vec{X}_{q+1} \ldots \vec{X}_{p+q}),$$

where $p+q = k$. Due to smoothness of the functionals, this is well defined and continuous, so this part yields a B-smooth functional;

case $J_2 \neq \emptyset$:

$$\begin{aligned}
&\int_{M^{k-2}} f_\varphi^{(|J_1|+1)}[0]_i\big(x, \vec{X}_{J_1}\big) \left(\mathcal{G}_\varphi^+\right)^{ij_1}(x, z_1)\delta^{(|I_1|+1)}E(\mathcal{L})_\varphi[0]_{j_1 j_2}\big(z_1, z_2, \vec{X}_{I_1}\big)\\
&\times \left(\mathcal{G}_\varphi^+\right)^{j_2 j_3}(z_2, z_3)\delta^{(|I_2|+1)}E(\mathcal{L})_\varphi[0]_{j_3 j_4}\big(z_3, z_4, \vec{X}_{I_2}\big) \cdots \delta^{(|I_l|+2)}E(\mathcal{L})_\varphi[0]_{j_{2l-1} j_{2l}}\big(z_{2l-1}, z_{2l}, \vec{X}_{I_l}\big)\\
&\times \left(\mathcal{G}_\varphi^+\right)^{j_{2l} j}(z_{2l}, y) g_\varphi^{(|J_3|+1)}[0]_j(y, \vec{X}_{p+k_1+\ldots+k_l+1}, \ldots, \vec{X}_{J_3}) d\mu_g(z_1, \ldots, z_{2l}),
\end{aligned}$$

where $I_1 \cup \ldots \cup I_l = J_2$. Again due to the B-smoothness of all functionals involved in the above formula, we conclude that the this piece too exists and is continuous. Hence as a whole $\mathsf{R}_\mathcal{L}(F,G)_\varphi^{(k)}$. Repeating the above calculations for $\mathsf{A}_\mathcal{L}$ amounts to substituting each $+$ with $-$, results as well in B-smoothness for the advanced product. Finally since $\{F,G\}_\mathcal{L} = \mathsf{R}_\mathcal{L}(F,G) - \mathsf{A}_\mathcal{L}(F,G)$ we conclude that it is smooth as well.

□

We have seen that the Peierls bracket is well defined for microlocal functionals, we stress however that the image under the Peierls bracket of microlocal functionals fails to be microlocal, it is therefore necessary to broaden the functional domain of this bracket. An idea is to use the full potential of microlocal analysis, and use wave front sets to define parings. First though we make explicit the "good" subset of T^*M, that is, those subsets in which the wavefront can be localized.

Definition 4.7. Let (M,g) be a Lorentzian spacetime, then define $\Upsilon_k(g) \subset T^*M^k$ as follows

$$\begin{aligned}
\Upsilon_k(g) \doteq \Big\{&(x_1, \ldots, x_k, \xi_1, \ldots, \xi_k) \in T^*M^k \backslash 0 :\\
&\qquad (x_1, \ldots, x_k, \xi_1, \ldots, \xi_k) \notin \overline{V}_k^+(x_1, \ldots, x_k) \cup \overline{V}_k^-(x_1, \ldots, x_k)\Big\}
\end{aligned} \tag{50}$$

where

$$\overline{V}_k^\pm(x_1, \ldots, x_k) = \prod_{j=1}^k \overline{V}^\pm(x_j).$$

recall that $V^\pm(x_j)$ are the cones of future/past directed vectors tangent at x_j. If $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ is open we say that a functional $F : \mathcal{U} \to \mathbb{R}$ with compact support is microcausal with respect to the Lorentz metric g in φ if $\mathrm{WF}(F_\varphi^{(k)}[0]) \cap \Upsilon_k(g) = \emptyset$ for all $k \in \mathbb{N}$. We say that F is microcausal with respect to g in $\mathcal{U}$ if F is microcausal for all $\varphi \in \mathcal{U}$. We denote the set of microcausal functionals in $\mathcal{U}$ by $\mathcal{F}_{\mu c}(B, \mathcal{U}, g)$.

One can show by induction, using (14), that the two definitions are equivalent. The case $k=1$ is trivial, while the case with arbitrary k follows from:

Lemma 4.8. *Suppose that for microcausal functional F there is a given symmetric linear connection having $\mathrm{WF}\left(\nabla^{(n-1)}F_\varphi[0]\right) \cap \Upsilon_{n-1}(g) = \emptyset$, then*

$$\mathrm{WF}\left(\nabla^{(n)}F_\varphi[0]\right) \cap \Upsilon_n(g) = \emptyset.$$

Proof. From (14) we have

$$\begin{aligned}
&\nabla^{(n)}F_\varphi[0](\vec{X}_1, \ldots, \vec{X}_n)\\
&\doteq F_\varphi^{(n)}[0](\vec{X}_1, \ldots, \vec{X}_n) + \sum_{j=1}^n \frac{1}{n!} \sum_{\sigma \in \mathcal{P}(n)} \nabla^{(n-1)}F_\varphi[0](\Gamma_\varphi(\vec{X}_{\sigma(j)}, \vec{X}_{\sigma(n)}), \vec{X}_{\sigma(1)}, \ldots, \widehat{\vec{X}_{\sigma(j)}}, \ldots \vec{X}_{\sigma(n-1)}) \,.
\end{aligned}$$

Assume that $\nabla^{(n-1)}F_\varphi[0]$ is microcausal. Since F is microcaulsal as well, it is sufficient to show microcausality holds for the other terms in the sum. Due to symmetry of the connetion, we can simply study the wave front set of a single term such as

$$\nabla^{(n-1)}F_\varphi(\Gamma_\varphi(\vec{X}_j, \vec{X}_n), \vec{X}_1, \ldots, \widehat{\vec{X}_j}, \ldots \vec{X}_{n-1}). \tag{51}$$

The idea is to apply Theorem 8.2.14 in [Hör15]. Recall that a connection Γ_φ can be seen as a mapping $\Gamma^\infty_c(M \leftarrow \varphi^*VB) \times \Gamma^\infty_c(M \leftarrow \varphi^*VB) \to \Gamma^\infty_c(M \leftarrow \varphi^*VB)$ with associated integral kernel $\Gamma[\varphi](x,y,z)$ defined by

$$\otimes^3\Gamma^\infty_c(M \leftarrow \varphi^*VB) \to \mathbb{R} : (\vec{X},\vec{Y},\vec{Z}) \mapsto \int_{M^3} h_{kl}(\varphi(x))\Gamma[\varphi]^l_{ij}(x,y,z)\vec{X}^i(x)\vec{Y}^j(y)\vec{Z}^k(z)d\mu_g(x,y,z)$$

where h is an auxiliary Riemannian metric on the fiber of the bundle B which is to be regarded as a tool for calculations. We can estimate the wave front set of $\Gamma[\varphi]^l_{ij}(x,y,z)$ by using the support properties of the connection coefficients Γ_φ and obtain $\Gamma[\varphi]^l_{ij}(x,y,z) = \Gamma^l_{ij}(\varphi(x))\delta(x,y,z)$, with $\Gamma^l_{ij}(\varphi(x))$ Christoffel coefficients of a connection on the typical fiber of B; thus

$$\mathrm{WF}\,(\Gamma[\varphi]) = \{(x,y,z,\xi,\eta,\zeta) \in T^*M^3\backslash 0 : x = y = z,\ \xi+\eta+\zeta = 0\}.$$

Composition of the two integral kernels in (51) is well defined provided $\mathrm{WF}'(\nabla^{(n-1)}F_\varphi[0])_M \cap \mathrm{WF}\,(\Gamma[\varphi])_M = \emptyset$ and that the projection map : $\triangle_3 M \to M$ is proper. The former is a consequence of $\mathrm{WF}\,(\Gamma[\varphi])_M = \emptyset$, the latter is a trivial statement for the diagonal embedding. Then we can apply Theorem 8.2.14, and estimate

$$\begin{aligned}
\mathrm{WF}\left(\nabla^{(n-1)}F_\varphi \circ \Gamma_\varphi\right) \subset & \Big\{(x_1,\dots,x_n,\xi_1,\dots,\xi_n) \in T^*M^n : \exists (y,\eta) :\ (x_j,x_n,y,\xi_j,\xi_n,-\eta) \in \mathrm{WF}\,(\Gamma[\varphi])\ , \\
& \quad (y,x_1,\dots,\widehat{x_j},\dots,x_{n-1},\eta,\xi_1,\dots,\widehat{\xi_j},\dots,\xi_{n-1}) \in \mathrm{WF}\left(\nabla^{(n-1)}F_\varphi[0]\right)\Big\} \\
& \bigcup \Big\{(x_1,\dots,x_n,\xi_1,\dots,\xi_n) \in T^*M^n : x_j = x_n,\ \xi_j = \xi_n = 0\ , \\
& \quad (y,x_1,\dots,\widehat{x_j},\dots,x_{n-1},0,\xi_1,\dots,\widehat{\xi_j},\dots,\xi_{n-1}) \in \mathrm{WF}\left(\nabla^{(n-1)}F_\varphi[0]\right)\Big\} \\
& \bigcup \Big\{(x_1,\dots,x_n,0,\dots,0,\xi_j,0,\dots,0,\xi_n) \in T^*M^n : (x_j,x_n,y,\xi_j,\xi_n,0) \in \mathrm{WF}\,(\Gamma_\varphi)\ , \\
& \quad (y,x_1,\dots,\widehat{x_j},\dots,x_{n-1},\eta,0,\dots,0) \in \mathrm{WF}\left(\nabla^{(n-1)}F_\varphi[0]\right)\Big\} \\
& = \Pi_1 \cup \Pi_2 \cup \Pi_3.
\end{aligned}$$

If by contradiction, we had that the $\nabla^{(n-1)}F_\varphi \circ \Gamma_\varphi$ was not microcausal, then there would be elements of its wavefront set for which all $\xi_1,\dots,\xi_n$ are, say, future pointing. In this case those must belong to Π_1, but then η is future pointing as well by the form of $\mathrm{WF}(\Gamma_\varphi)$, so that $\nabla^{(n-1)}F_\varphi$ is not microcausal, contradicting our initial assumption. □

One can also show that microcausality does not depend upon the connection chosen by computing

$$\begin{aligned}
&\nabla^{(n)}F_\varphi[0](\vec{X}_1,\dots,\vec{X}_n) - \widetilde{\nabla}^{(n)}F_\varphi[0]\big(\vec{X}_1,\dots,\vec{X}_n\big) \\
&\qquad = \sum_{j=1}^n \frac{1}{n!}\sum_{\sigma\in\mathcal{P}(n)} \nabla^{(n-1)}F_\varphi[0]\Big(\Gamma_\varphi\big(\vec{X}_{\sigma(j)},\vec{X}_{\sigma(n)}\big),\vec{X}_{\sigma(1)},\dots,\widehat{\vec{X}_{\sigma(j)}},\dots\vec{X}_{\sigma(n-1)}\Big) \\
&\qquad - \sum_{j=1}^n \frac{1}{n!}\sum_{\sigma\in\mathcal{P}(n)} \widetilde{\nabla}^{(n-1)}F_\varphi[0]\Big(\widetilde{\Gamma}_\varphi\big(\vec{X}_{\sigma(j)},\vec{X}_{\sigma(n)}\big),\vec{X}_{\sigma(1)},\dots,\widehat{\vec{X}_{\sigma(j)}},\dots\vec{X}_{\sigma(n-1)}\Big);
\end{aligned}$$

and then combining induction with Lemma 4.8 to get an empty wave front set for the terms on right hand side of the above equation. Another consequence of Lemma 4.8 is that microcausality of a functional does not depend on the Γ^∞-local charts used to perform the derivatives. We immediately have the inclusion $\mathcal{F}_{reg}(B,\mathcal{U}) \subset \mathcal{F}_{\mu c}(B,\mathcal{U},g)$.

Proposition 4.9. *Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open, then if $F \in \mathcal{F}_{\mu loc}(B,\mathcal{U})$, $\mathrm{WF}\big(F^{(k)}_\varphi[0]\big)$ is conormal to $\triangle_k(M)$ i.e. $\mathrm{WF}\left(F^{(k)}_\varphi[0]\right) \subset \{(x,\dots,x,\xi_1,\dots,\xi_k) : \xi_1+\dots+\xi_k = 0\}$ for all $k \geq 2$ and $\varphi \in \mathcal{U}$. Therefore $\mathcal{F}_{\mu loc}(B,\mathcal{U}) \subset \mathcal{F}_{\mu c}(B,\mathcal{U},g)$.*

Proof. Since the wavefront set is a local property independent from the chart, we fix any $\mathcal{U}_\varphi$ and calculate it there. Note that the first derivative results in a smooth functional, then we take the kth derivative with $k \geq 2$. Going through the calculations, we get that $F^{(k)}_\varphi[0]\big(\vec{X}_1,\dots,\vec{X}_k\big)$ defines an integral kernel of the form

$$\int_{M^k} f^{(k)}_\varphi[0]_{i_1\cdots i_k}(x_1)\delta(x_1,\dots,x_k)\vec{X}^{i_1}_1(x_1)\cdots\vec{X}^{i_k}_k(x_k)d\mu_g(x_1,\dots,x_k)\ .$$

where $f^{(k)}_\varphi[0]_{i_1\cdots i_k}$ is some smooth function for each indices $i_1,\dots,i_k$. The calculation of the wavefront of such an integral kernel is equivalent to the calculation of the wave front of the diagonal delta, resulting in a subset of the normal bundle to the diagonal map image. Therefore

$$\mathrm{WF}(F_\varphi^{(k)}[0]) = N^*\triangle_k(M) = \left\{(x_1,\dots,x_k,\xi_1,\dots,\xi_k)\in T^*M^k\backslash 0:\ x_1=\dots=x_k;\ \sum_{j=1}^k \xi_j = 0\right\}.$$

In addition if $(x,\dots,x,\xi_1,\dots,\xi_k)$ is in $\mathrm{WF}\big(F_\varphi^{(k)}[0]\big)$ and has, say, the first $k-1$ covectors in $\overline{V}^+_{k-1}(x,\dots,x)$, by conormality $\xi_k = -(\xi_1+\ldots+\xi_{k-1})$ and we see that $\xi_k\in\overline{V}^-(x)$, whence microlocality implies microcausality. □

Theorem 4.10. *Let $\mathcal{U}\subset\Gamma^\infty(M\leftarrow B)$ be CO-open and $\mathcal{L}$ a generalized microlocal Lagrangian with normally hyperbolic linearized equations. Then the Peierls bracket associated to $\mathcal{L}$ extends to $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$, has the same support property of Proposition 4.6 and depends only locally on $\mathcal{L}$, that is, for all F, $G\in\mathcal{F}_{\mu c}(B,\mathcal{U},g)$, $\{F,G\}_{\mathcal{L}}$ is unaffected by perturbations of $\mathcal{L}$ outside the right hand side of (47). The same locality property holds for the retarded and advanced products.*

Proof. Clearly $\{F,G\}_{\mathcal{L}}$ is well defined, in fact since $\mathrm{WF}(G_\varphi^{(1)}[0])$ is spacelike, and $\mathcal{G}_\varphi$ according to Theorem 4.2 propagates only lightlike singularities along lightlike geodesics, then $\mathcal{G}_\varphi G_\varphi^{(1)}[0]$ must be smooth, giving then a well defined pairing. As for support properties the proof can be carried on analogously to the proof of Proposition 4.6.

We now study the local behavior of the bracket. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are generalized Lagrangians, such that for some fixed $\varphi\in\mathcal{U}$, $\delta^{(1)}E(\mathcal{L}_1)_\varphi[0]$ and $\delta^{(1)}E(\mathcal{L}_2)\varphi[0]$ differ only in a region outside

$$\mathcal{O}\doteq\big(J^+(\mathrm{supp}(F))\cup J^-(\mathrm{supp}(F))\big)\cap\big(J^+(\mathrm{supp}(G))\cup J^-(\mathrm{supp}(G))\big). \tag{52}$$

By the support properties of retarded and advanced propagators of Proposition 4.6 we have

$$\left\langle F_\varphi^{(1)}[0], (\mathcal{G}^+_{\varphi,\mathcal{L}_1}-\mathcal{G}^+_{\varphi,\mathcal{L}_2})G_\varphi^{(1)}[0]\right\rangle = 0,$$

as well as

$$\left\langle F_\varphi^{(1)}[0], (-\mathcal{G}^-_{\varphi,\mathcal{L}_1}+\mathcal{G}^-_{\varphi,\mathcal{L}_2})G_\varphi^{(1)}[0]\right\rangle = 0.$$

Taking the sum of the two we find

$$\{F,G\}_{\mathcal{L}_1}-\{F,G\}_{\mathcal{L}_2}=0.$$

□

Theorem 4.11. *Let $\mathcal{U}\subset\Gamma^\infty(M\leftarrow B)$ CO-open and $\mathcal{L}$ a generalized Lagrangian. If F, $G\in\mathcal{F}_{\mu c}(B,\mathcal{U},g)$ we have that $\{F,G\}_{\mathcal{L}}\in\mathcal{F}_{\mu c}(B,\mathcal{U},g)$ as well.*

Proof. By Faà di Bruno's formula,

$$\begin{aligned}&\{F,G\}^{(k)}_{\mathcal{L},\varphi}[0](\vec{X}_1,\dots,\vec{X}_k)\\&\quad=\sum_{(J_1,J_2,J_3)\subset P_k}\left\langle F_\varphi^{(|J_1|+1)}[0](\otimes_{j_1\in J_1}\vec{X}_{j_1})\ ,\ d^{(|J_2|)}\mathcal{G}_\varphi(\otimes_{j_2\in J_2}\vec{X}_{j_2})G_\varphi^{(|J_3|+1)}[0](\otimes_{j_3\in J_3}\vec{X}_{j_3})\right\rangle.\end{aligned} \tag{53}$$

while by (40),

$$\begin{aligned}&d^{|J_2|}\mathcal{G}_\varphi(\vec{X}_1,\dots,\vec{X}_k)\\&\quad=\sum_{l=1}^k(-1)^l\sum_{\substack{(I_1,\dots,I_l)\\\in\mathcal{P}(J_2)}}\sum_{p=0}^l\left(\bigcirc_{i=1}^p\,\mathcal{G}^-_\varphi\circ\delta^{(|I_i|+1)}E(\mathcal{L})\big(\vec{X}_{I_i}\big)\right)\circ\mathcal{G}_\varphi\circ\left(\bigcirc_{i=p+1}^l\,\delta^{(|I_i|+1)}E(\mathcal{L})_\varphi\big(\vec{X}_{I_i}\big)\circ\mathcal{G}^+_\varphi\right),\end{aligned} \tag{54}$$

where $\bigcirc_{i=1}^p$ stands for composition of mappings indexed by i from 1 to p. For the rest of the proof, we will use the integral notation we used in (15) and in the proof of Proposition 4.6. Recall that, by (29), the mapping $\delta^{(n)}E(\mathcal{L})_\varphi[0]$ has associated a compactly supported integral kernel $\mathcal{L}(1)^{(n+1)}_\varphi[0](x,z_1,\dots,z_n)$ and its wave front is in $N^*\triangle_{n+1}(M)$ by Proposition 4.9. Then again, we have two general cases:
1) $J_2=\emptyset$.
Then, letting $|J_1|=p$, $|J_3|=q=k-p$, the typical term has the form

$$\{F,G\}^{(k)}_\varphi[0](z_1,\dots,z_k)=\int_{M^2}f_\varphi^{(p+1)}[0]_i(x,z_1,\dots,z_p),\mathcal{G}^{ij}_\varphi(x,y)g_\varphi^{(q+1)}[0]_j(y,z_{p+1},\dots,z_k)d\mu_g(x,y). \tag{55}$$

Suppose by contradiction that there is some $(x_1,\dots,x_k,\xi_1,\dots,\xi_k) \in \mathrm{WF}(\{F,G\}_\varphi[0])$ has $(\xi_1,\dots,\xi_k) \in \overline{V}_k^+(x_1,\dots,x_k)$ (the argument works similarly for $(\xi_1,\dots,\xi_k) \in \overline{V}_k^-(x_1,\dots,x_k)$). Using twice Theorem 8.2.14 in [Hör15] in the above pairing yields

$$
\begin{aligned}
&\mathrm{WF}\Big(\{F,G\}_\varphi^{(k)}[0]\Big)\\
&\quad\subseteq \Big\{(z_1,\dots,z_k,\xi_1,\dots,\xi_k):\ \exists (y,\eta)\in T^*M (x,z_1,\dots,z_p,-\eta,\xi_1,\dots,\xi_p)\in \mathrm{WF}(F_\varphi^{(p+1)}[0]_i),\\
&\qquad (x,z_{p+1},\dots,z_k,\eta,\xi_{p+1},\dots,\xi_k)\in \mathrm{WF}\big(\mathcal{G}_\varphi^{ij}G_\varphi^{(q+1)}[0]_j\big)\Big\}\\
&\quad\subset \Big\{(z_1,\dots,z_k,\xi_1,\dots,\xi_k):\ \exists (x,\eta),(y,\zeta)\in T^*M : (x,z_1,\dots,z_p,-\eta,\xi_1,\dots,\xi_p)\in \mathrm{WF}\big(F_\varphi^{(p+1)}[0]_i\big)\\
&\qquad (x,y,\eta,-\zeta)\in \mathrm{WF}(\mathcal{G}_\varphi^{ij}),\ (y,z_{p+1},\dots,z_k,\zeta,\xi_{p+1},\dots,\xi_k)\in \mathrm{WF}(G_\varphi^{(q+1)}[0]_j)\Big\}.
\end{aligned}
$$

So if $(z_1,\dots,z_k,\xi_1,\dots,\xi_k)\in \mathrm{WF}\big(\{F,G\}_\varphi^{(k)}\big)$, then $\exists\ (x,\eta),(y,\zeta)\in T^*M$ such that

$$
\begin{cases}
(x,z_1,\dots,z_p,-\eta,\xi_1,\dots,\xi_p) & \in \mathrm{WF}(F_\varphi^{(p+1)}[0]_i))\\
(x,y,\eta,-\zeta) & \in \mathrm{WF}(\mathcal{G}_\varphi^{ij})\\
(y,z_{p+1},\dots,z_k,\zeta,\xi_{p+1},\dots,\xi_k) & \in \mathrm{WF}(G_\varphi^{(q+1)}[0]_j)\ .
\end{cases}
$$

Now by Theorem 4.2, $\mathrm{WF}(\mathcal{G}_\varphi^{ij})$ contains pairs of lightlike covectors with opposite time orientation, therefore in case $\eta\in\overline{V}^+(x)$ (resp. $\eta\in\overline{V}^-(x)$), then $\zeta\in\overline{V}^+(y)$ (resp. $\zeta\in\overline{V}^-(y)$) in which case $\mathrm{WF}\big(G_\varphi^{(q+1)}[0]_j\big)$ (resp. $\mathrm{WF}\big(F_\varphi^{(p+1)}[0]_i\big)$) does violate the microcausality condition of Definition 4.7.

2) $J_2\neq\emptyset$.

Again let $|J_1|=p$, $|J_3|=k-q$, set also, referring to (54), $|I_j|=k_j$ for $j=1,\dots,l$ so that $|J_2|=k_1+\dots+k_l$. Combining (53) with (54) with the integral kernel notation we get

$$
\begin{aligned}
\{F,G\}_\varphi^{(k)}[0](z_1,\dots,z_k) = \int_{M^k} & f_\varphi^{(p+1)}[0]_i(x,z_1,\dots,z_p)\mathcal{G}_\varphi^{-\ ij_1}(x,x_1)d^{(k_1+2)}\mathcal{L}_\varphi[0]_{j_1i_1}(x_1,y_1,z_{I_1})\\
&\times \mathcal{G}_\varphi^{-\ i_1j_2}(y_1,x_2)\cdots\mathcal{G}_\varphi^{-\ i_{m-1}j_m}(y_{m-1},x_m)d^{(k_m+2)}\mathcal{L}_\varphi[0]_{j_mi_m}(x_m,y_m,z_{I_m})\\
&\times \mathcal{G}_\varphi^{i_mj_{m+1}}(y_m,x_{m+1})d^{(k_{m+1}+2)}\mathcal{L}_\varphi[0]_{j_{m+1}i_{m+1}}(x_{m+1},y_{m+1},z_{I_{m+1}})\\
&\times \mathcal{G}_\varphi^{+\ i_{m+1}j_{m+2}}(y_{m+1},z_{m+2})\dots d^{(k_l+2)}\mathcal{L}_\varphi[0]_{j_li_l}(x_l,y_l,z_{I_l},)\mathcal{G}^{+\ i_lj}(y_l,y)\\
&\times g_\varphi^{(k-q+1)}[0]_j(y,z_{q+1},\dots,z_k)d\mu_g(x,x_1,y_1,\dots,x_l,y_l,y).
\end{aligned}
\tag{56}
$$

Combining Theorem 8.2.14 in [Hör15], Theorem 4.2 and Proposition 4.9 we can estimate the wave front set of the integral kernel of $\{F,G\}_\varphi^{(k)}[0]$ as all elements $(z_1,\dots,z_k,\xi_1,\dots,\xi_k)\in T^*M^k$ for which there are (x,η), (x_1,η_1), $\dots$, $(x_l\eta_l)$, (y_1,ζ_1), $\dots$, (y_l,ζ_l) $(y,\zeta)\in T^*M$ such that

$$
\begin{cases}
(x,z_1,\dots,z_p,-\eta,\xi_1,\dots,\xi_p) & \in \mathrm{WF}\big(f_\varphi^{(p+1)}[0]_i\big),\\
(x,x_1,\eta,-\eta_1) & \in \mathrm{WF}(\mathcal{G}_\varphi^{-\ ij_1}),\\
(x_1,y_1,z_{|I_1|},\eta_1,-\zeta_1,\xi_{I_1}) & \in \mathrm{WF}\big(d^{(k_1+2)}\mathcal{L}_\varphi[0]_{i_1j_1}\big),\\
\vdots & \vdots\\
(y_{m-1},x_m,\zeta_{m-1},-\eta_m) & \in \mathrm{WF}(\mathcal{G}_\varphi^{-\ i_{m-1}j_m}),\\
(x_m,y_m,z_{I_m},\eta_m,-\zeta_m,\xi_{I_m}) & \in \mathrm{WF}\big(d^{(k_m+2)}\mathcal{L}_\varphi[0]_{i_mj_m}\big),\\
(y_m,x_{m+1},\zeta_m,-\eta_{m+1}) & \in \mathrm{WF}(\mathcal{G}_\varphi^{i_mj_{m+1}}),\\
(x_{m+1},y_{m+1},z_{I_{m+1}},\eta_{m+1},-\zeta_{m+1},\xi_{I_{m+1}}) & \in \mathrm{WF}\big(d^{(k_{m+1}+2)}\mathcal{L}_\varphi[0]_{i_{m+1}j_{m+1}}\big),\\
(y_{m+1},x_{m+2},\zeta_{m+1},-\eta_{m+2}) & \in \mathrm{WF}(\mathcal{G}_\varphi^{+\ i_{m+1}j_{m+2}}),\\
\vdots & \vdots\\
(y_{l-1},x_l,\zeta_{l-1},-\eta_l) & \in \mathrm{WF}(\mathcal{G}_\varphi^{+\ i_{l-1}j_l}),\\
(x_l,y_l,z_{I_l},\eta_l,-\zeta_l,\xi_{I_l}) & \in \mathrm{WF}\big(d^{(k_l+2)}\mathcal{L}_\varphi[0]_{i_lj_l}\big),\\
(y_l,y,\zeta_l,-\zeta) & \in \mathrm{WF}(\mathcal{G}_\varphi^{+\ i_lj}),\\
(y,z_{k-q+1},\dots,z_k,\zeta,\xi_{k-q+1},\dots,\xi_k) & \in \mathrm{WF}\big(g_\varphi^{(k-q+1)}[0]_j\big).
\end{cases}
$$

Suppose by contradiction, as above, that $(z_1,\dots,z_k,\xi_1,\dots,\xi_k)\in \mathrm{WF}\big(\{F,G\}_\varphi^{(k)}\big)$ has $(\xi_1,\dots,\xi_k)\in\overline{V}_k^+(z_1,\dots,z_k)$ $\big($resp. $(\xi_1,\dots,\xi_k)\in\overline{V}_k^-(z_1,\dots,z_k)\big)$. Then ζ_m and η_{m+1} are both either lightlike future directed, or lightlike past directed. In the first case, propagation of singularities implies that ζ is lightlike future directed, contradicting microcausality of $G_\varphi^{(k-q+1)}[0]$ (resp. ζ is lightlike past directed, contradicting the microlocality of $G_\varphi^{(k-q+1)}[0]$); in the second case, propagation of singularities implies that η is lightlike past directed, contradicting microcausality of $F_\varphi^{(p+1)}[0]$ (resp. η is lightlike future directed, contradicting the microlocality of $F_\varphi^{(p+1)}[0]$). We remark that in the wave front set of $\{F,G\}_\varphi^{(k)}[0]$ is the (finite) union under all possible choices of indices for all wave front sets of the form (55) or (56), each of which is however microcausal, implying that their finite union will be microcausal as well. ☐

Theorem 4.12. *The mapping $(F,G)\mapsto\{F,G\}_{\mathcal{L}}$ defines a Lie bracket on $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$, for all $\varphi\in\mathcal{U}$ CO-open.*

Proof. Bilinearity and antisymmetry are clear from Definition 4.5, while Theorem 4.11 ensures the closure of the bracket operation. We are thus left with the Jacobi identity:

$$\{F,\{G,H\}_{\mathcal{L}}\}_{\mathcal{L}}+\{G,\{H,F\}_{\mathcal{L}}\}_{\mathcal{L}}+\{H,\{F,G\}_{\mathcal{L}}\}_{\mathcal{L}}=0.$$

Using the integral kernel notation as in the above proof, we have

$$\begin{aligned}
&\{F,\{G,H\}_{\mathcal{L}}\}_{\mathcal{L}}(\varphi)=\int_{M^2} f_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)\{G,H\}^{(1)}_{\mathcal{L}\ \varphi}[0]_j(y)d\mu_g(x,y)\\
&=\int_{M^4} f_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)\left(g_\varphi^{(2)}[0]_{jk}(y,z)\mathcal{G}_\varphi^{kl}(z,w)h_\varphi^{(1)}[0]_l(w)+g_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{kl}(z,w)h_\varphi^{(2)}[0]_{jl}(y,w)\right)d\mu_g(x,y,z,w)\\
&\quad-\int_{M^6} f_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)\Big(d^{(3)}\mathcal{L}_\varphi[0]_{jj_1i_1}(y,y_1,x_1,)\mathcal{G}_\varphi^{-\ kj_1}(z,y_1)g_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{i_1l}(x_1,w)h_\varphi^{(1)}[0]_l(w)\\
&\qquad+d^{(3)}\mathcal{L}_\varphi[0]_{jj_1i_1}(y_1,x_1,y)\mathcal{G}_\varphi^{kj_1}(z,y_1)g_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{+\ i_1l}(x_1,w)h_\varphi^{(1)}[0]_l(w)\Big)d\mu_g(x,y,z,w,x_1,y_1).
\end{aligned}$$

Summing over cyclic permutations of the first two terms yields

$$\begin{aligned}
&\int_{M^4}\Big(F_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)G_\varphi^{(2)}[0]_{jk}(y,z)\mathcal{G}_\varphi^{kl}(z,w)H_\varphi^{(1)}[0]_l(w)\\
&\quad+F_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)G_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{kl}(z,w)H_\varphi^{(2)}[0]_{jl}(y,w)\\
&\quad+G_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)H_\varphi^{(2)}[0]_{jk}(y,z)\mathcal{G}_\varphi^{kl}(z,w)F_\varphi^{(1)}[0]_l(w)\\
&\quad+G_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)H_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{kl}(z,w)F_\varphi^{(2)}[0]_{jl}(y,w)\\
&\quad+H_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)F_\varphi^{(2)}[0]_{jk}(y,z)\mathcal{G}_\varphi^{kl}(z,w)G_\varphi^{(1)}[0]_l(w)\\
&\quad+H_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)F_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{kl}(z,w)G_\varphi^{(2)}[0]_{jl}(y,w)\Big)d\mu_g(x,y,z,w)\\
&\quad=0,
\end{aligned}$$

while for the other two,

$$\begin{aligned}
&\int_{M^6}d\mu_g(x,y,z,w,x_1,y_1)\\
&\Big(f_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)d^{(3)}\mathcal{L}_\varphi[0]_{jj_1x_1}(y,y_1,x_1,)\mathcal{G}_\varphi^{-\ kj_1}(z,y_1)g_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{i_1l}(x_1,w)h_\varphi^{(1)}[0]_l(w)\\
&+f_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)d^{(3)}\mathcal{L}_\varphi[0]_{jj_1i_1}(y,y_1,x_1)\mathcal{G}_\varphi^{kj_1}(z,y_1)g_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{+\ i_1l}(x_1,w)h_\varphi^{(1)}[0]_l(w)\\
&+g_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)d^{(3)}\mathcal{L}_\varphi[0]_{jj_1i_1}(y,y_1,x_1,)\mathcal{G}_\varphi^{-\ kj_1}(z,y_1)h_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{i_1l}(x_1,w)f_\varphi^{(1)}[0]_l(w)\\
&+g_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)d^{(3)}\mathcal{L}_\varphi[0]_{jj_1i_1}(y,y_1,x_1)\mathcal{G}_\varphi^{kj_1}(z,y_1)h_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{+\ i_1l}(x_1,w)f_\varphi^{(1)}[0]_l(w)\\
&+h_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)d^{(3)}\mathcal{L}_\varphi[0]_{jj_1i_1}(y,y_1,x_1,)\mathcal{G}_\varphi^{-\ kj_1}(z,y_1)f_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{i_1l}(x_1,w)g_\varphi^{(1)}[0]_l(w)\\
&+h_\varphi^{(1)}[0]_i(x)\mathcal{G}_\varphi^{ij}(x,y)d^{(3)}\mathcal{L}_\varphi[0]_{jj_1i_1}(y,y_1,x_1)\mathcal{G}_\varphi^{kj_1}(z,y_1)f_\varphi^{(1)}[0]_k(z)\mathcal{G}_\varphi^{+\ i_1l}(x_1,w)g_\varphi^{(1)}[0]_l(w)\Big)\\
&=0.
\end{aligned}$$

To make the simplifications we used the antisymmetry of the integral kernel $\mathcal{G}_\varphi(x,y)$, the adjoint relation between the propagators $\mathcal{G}^+(x,y)=\mathcal{G}^-(y,x)$ (see Lemma 4.4) and $\mathcal{G}_\varphi^{ij}=\mathcal{G}_\varphi^{ji}$. ☐

5. Structure of the space of microcausal functionals

The first point of emphasis is to give a topology to $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$. We shall proceed step by step refining our starting definitions to better grasp the reasoning behind the topology we will end up giving $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$.

The simplest guess, as well as the weakest, on $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$ is the locally convex topology that corresponds to the initial topology induced by the mappings

$$F \to F(\varphi) \in \mathbb{C}.$$

To account for smooth functionals we try the initial topology with respect to mappings

$$\begin{aligned} &F \to F(\varphi) \in \mathbb{C}, \\ &F \to \nabla^{(k)} F_\varphi[0] \in \Gamma^{-\infty}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right). \end{aligned}$$

This time we are leaving out all information on the wave front set which plays a role in defining microcausal functionals. To remedy we would like to set up the Hörmander topology on the spaces $\Gamma^{-\infty}_{\Upsilon_{k,g}}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right)$, however this is not immediately possible since $\Upsilon_{k,g}$ are open cones, and the Hörmander topology is given to closed ones, therefore we need the following result, whose proof can be found in Lemma 4.1 in [BFR19],

Lemma 5.1. *Given the open cone $\Upsilon_k(g)$ it is always possible to find a sequence of closed cones $\{\mathcal{V}_m(k) \subset T^*M^k\}_{m\in\mathbb{N}}$ such that $\mathcal{V}_m(k) \subset \mathrm{Int}(\mathcal{V}_{m+1}(k))$ and $\cup_{m\in\mathbb{N}}\mathcal{V}_m(k) = \Upsilon_k(g)$ for all $k \geq 1$.*

Then we can write

$$\Gamma^{-\infty}_{\Upsilon_k(g)}\left(M^k \leftarrow \boxtimes_k \varphi^* VB\right) = \varinjlim_{m\in\mathbb{N}} \Gamma^{-\infty}_{\mathcal{V}_m(k)}\left(M^k \leftarrow \boxtimes_k \varphi^* VB\right). \tag{57}$$

By construction of the direct limit we have mappings

$$\Gamma^{-\infty}_{\mathcal{V}_m(k)}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right) \to \Gamma^{-\infty}_{\Upsilon_k(g)}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right)$$

where the source space has $\mathcal{V}_m(k) \subset T^*M^k$ as a closed cone, so it can be given the Hörmander topology. In particular when we are dealing with standard compactly supported distributions its topology can be defined, see the remark after Theorem 18.1.28 in [Hör07], to be the initial topology with respect to the mappings

$$\begin{aligned} &F \to F(\varphi) \in \mathbb{C}, \\ &F \to PF \in \Gamma^\infty_c\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right) \end{aligned}$$

where φ is any smooth section of B and P any properly supported pseudodifferential operator of order zero on the vector bundle $\boxtimes^k (\varphi^* VB) \to M^k$ such that $\mathrm{WF}(P) \cap \mathcal{V}_m(k) = \emptyset$. Using that Definition 18.1.32 [Hör07], Theorem 18.1.16 [Hör07] and Theorem 8.2.13 in [Hör15] can be generalized to the vector bundle case provided we use the notion (32) for the wave front set of vector valued distributions; we can argue as in Corollary 4.1 of [BFR19] that each $\Gamma^{-\infty}_{\mathcal{V}_m(k)}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right)$ becomes a Hausdorff topological space. By Lemma 30.4 pp. 296-297 in [KM97] since the base manifold M is separable and the fibers are finite dimensional vector spaces, hence nuclear[8] and Fréchet, we find that $\Gamma^\infty_c\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right)$ is a nuclear limit-Fréchet space, it is Hausdorff, thus each

$$\Gamma^{-\infty}_{\mathcal{V}_m(k)}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right)$$

is nuclear as well. Finally by Porposition 50.1 pp. 514 in [Tre16] the direct limit topology on

$$\Gamma^{-\infty}_{\Upsilon_{k,g}}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right)$$

is nuclear for all k (and also Hausdorff). We have therefore proved:

Lemma 5.2. *The direct limit topology on $\Gamma^{-\infty}_{\Upsilon_{k,g}}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right)$ induced as a direct limit topology of the spaces $\Gamma^{-\infty}_{\mathcal{V}_m(k)}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right)$ with the Hörmander topology is a Hausdorff nuclear space.*

Finally we can induce on $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$ a topology by

Theorem 5.3. *Given the set $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$, consider the mappings*

$$\mathcal{F}_{\mu c}(B,\mathcal{U},g) \ni F \mapsto F(\varphi) \in \mathbb{C}, \tag{58}$$

$$\mathcal{F}_{\mu c}(B,\mathcal{U},g) \ni F \mapsto \nabla^{(k)} F_\varphi[0] \in \Gamma^{-\infty}_{\Upsilon_{k,g}}\left(M^k \leftarrow \boxtimes^k (\varphi^* VB)\right), \tag{59}$$

*and the related initial topology on $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$. Then $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$ is a nuclear locally convex topological space with a Poisson *-algebra with respect to the Peierls bracket of some microlocal generalized Lagrangian $\mathcal{L}$.*

[8]We say that a Hausdorff locally convex space E is nuclear if given any other locally convex space F we have $E \otimes_\pi F \simeq E \otimes_\epsilon F$, where the two are the tensor product space respectively endowded with the quotient topology and with the canonical topology associated to the space of continuous bilinear mappings : $E'_\sigma \times F'_\sigma \to \mathbb{R}$ equpped with the topology of uniform convergence on products of equicontinuous subsets of E' and F'. For more detail see either [Pie22] or [Tre16].

Proof. The nuclearity follows from the stability of nuclear spaces under projective limit topology, see Proposition 50.1 pp. 514 in [Tre16], so using the nuclearity of both $\Gamma^{-\infty}_{\Upsilon_{k,g}}\left(M^k \leftarrow \boxtimes^k(\varphi^* VB)\right)$ (via Lemma 5.2) and $\mathbb{C}$ (trivially) we have our claim. The Peierls bracket is well defined by Theorem 4.11 and satisfies the Jacoby identity due to Theorem 4.12, so we only have to show the Leibniz rule for the braket, that is

$$\{F, GH\}_{\mathcal{L}} = G\{F, H\}_{\mathcal{L}} + \{F, G\}_{\mathcal{L}} H.$$

However this follows once we show that the product : $F, G \mapsto F \cdot G$ with $(F \cdot G)(\varphi) = F(\varphi)G(\varphi)$ is closed in $\mathcal{F}_{\mu c}(B, \mathcal{U}, g)$ and then use $(F\cdot G)^{(1)}_\varphi[0] = F^{(1)}_\varphi[0]G(\varphi) + F(\varphi)G^{(1)}_\varphi[0]$. The latter is a consequence of the definition of derivation (i.e. the standard Leibniz rule), so we are left with showing the former: we compute

$$(F \cdot G)^{(k)}_\varphi[0](\vec{X}_1, \ldots, \vec{X}_k) = \sum_{\sigma\in\mathcal{P}(1,\ldots,k)} \sum_{l=0}^{k} F^{(l)}_\varphi[0](\vec{X}_{\sigma(1)}, \ldots, \vec{X}_{\sigma(l)}) G^{(k-l)}_\varphi[0](\vec{X}_{\sigma(k-l+1)}, \ldots, \vec{X}_{\sigma(k)}),$$

where $\mathcal{P}(1, \ldots, k)$ is the set of permutations of $\{1, \ldots, k\}$. For each of those terms using Theorem 8.2.9 in [Hör15] we have

$$\begin{aligned} \mathrm{WF}(F^{(l)}_\varphi[0]G^{(k-l)}_\varphi[0]) \subset\ & \mathrm{WF}(F^{(l)}_\varphi[0]) \times \mathrm{WF}(G^{(k-l)}_\varphi[0]) \\ & \bigcup \mathrm{WF}(G^{(k-l)}_\varphi[0]) \times \Big(\mathrm{supp}(G^{(k-l)}_\varphi[0]) \times \{0\}\Big) \\ & \bigcup \Big(\mathrm{supp}(F^{(l)}_\varphi[0]) \times \{0\}\Big) \times \mathrm{WF}(G^{(k-l)}_\varphi[0]), \end{aligned}$$

and therefore microcausality is met. □

Note that closed linear subspaces of $\mathcal{F}_{\mu c}(B, \mathcal{U}, g)$ are nuclear as well (see Proposition 50.1 in [Tre16]), so $\mathcal{F}_{\mu loc}(B, \mathcal{U}, g)$ is a Hausdorff nuclear space. The space $\mathcal{F}_{\mu c}(B, \mathcal{U}, g)$ can be given a structure of a C^∞-ring (see [MR13] for details), more precisely

Proposition 5.4. *If $F_1, \ldots, F_n \in \mathcal{F}_{\mu c}(B, \mathcal{U}, g)$ and $\psi \in V \subset \mathbb{C}^n \to \mathbb{C}$ is smooth, then $\psi(F_1, \ldots, F_n) \in \mathcal{F}_{\mu c}(B, \mathcal{U}, g)$ and*

$$\mathrm{supp}(\psi(F_1, \ldots, F_n)) \subset \bigcup_{i=1}^{n} \mathrm{supp}(F_i).$$

Proof. First we check the support properties. Suppose that $x \notin \cup_{i=1}^n \mathrm{supp}(F_i)$, we can find an open neighborhood V of x for which given any $\varphi \in \mathcal{U}$ and any $\vec{X} \in \Gamma^\infty_c(M \leftarrow \varphi^* VB)$ having $\mathrm{supp}(\vec{X}) \subset V$ implies $(F_i \circ u_\varphi)(t\vec{X}) = (F_i \circ u_\varphi)(0)$ for all t in a suitable neighborhood of $0 \in \mathbb{R}$. Then $\psi\big((F_1 \circ u_\varphi)(t\vec{X}), \ldots, (F_n \circ u_\varphi)(t\vec{X})\big) = \psi\big((F_1 \circ u_\varphi)(0), \ldots, (F_n \circ u_\varphi)(0)\big)$ as well giving $x \notin \mathrm{supp}(\psi \circ (F_1, \ldots, F_n))$. We immediately see that the composition is B-smooth, so consider its kth derivative expressed via Faà di Bruno's formula:

$$\begin{aligned} &\psi(F_1, \ldots, F_1)^{(k)}_\varphi[0](\vec{X}_1, \ldots, \vec{X}_k) \\ &= \sum_{(J_1,\ldots,J_n)\in\mathcal{P}(1,\ldots,k)} \frac{\partial^k \psi(F_1(\varphi), \ldots, F_n(\varphi))}{\partial \mathsf{Re}(z)^{J_1+\cdots+J_n}} \Big(\mathsf{Re} F^{(|J_1|)}_{1\ \varphi}[0](\vec{X}_{j_{1,1}}, \ldots, \vec{X}_{j_{|J_1|,1}}) \cdot \ldots \cdot \mathsf{Re} F^{(|J_n|)}_{n\ \varphi}[0](\vec{X}_{j_{1,n}}, \ldots, \vec{X}_{j_{|J_n|,n}})\Big) \\ &+ \frac{\partial^k \psi(F_1(\varphi), \ldots, F_n(\varphi))}{\partial \mathsf{Im}(z)^{J_1+\cdots+J_n}} \Big(\mathsf{Im} F^{(|J_1|)}_{1\ \varphi}[0](\vec{X}_{j_{1,1}}, \ldots, \vec{X}_{j_{|J_1|,1}}) \cdot \ldots \cdot \mathsf{Im} F^{(|J_n|)}_{n\ \varphi}[0](\vec{X}_{j_{1,n}}, \ldots, \vec{X}_{j_{|J_n|,n}})\Big) \end{aligned}$$

where Re and Im denote the real and imaginary parts of complex elements. Since ψ is smooth the only contribution to the wavefront set of the composition is the product of functional derivative in the above sum, for which Theorem 8.2.9 in [Hör15] gives

$$\begin{aligned} \mathrm{WF}\left(F^{(|J_1|)}_1, \ldots, F^{(|J_n|)}_n\right) \subset\ & \mathrm{WF}\left(F^{(|J_1|)}_{1\ \varphi}\right) \times \ldots \times \mathrm{WF}\left(F^{(|J_n|)}_{n\ \varphi}\right) \\ & \bigcup \mathrm{supp}\left(F^{(|J_1|)}_{1\ \varphi}\right) \times \{\vec{0}\}^{|J_1|} \times \mathrm{WF}\left(F^{(|J_2|)}_{2\ \varphi}\right) \times \ldots \times \mathrm{WF}\left(F^{(|J_n|)}_{n\ \varphi}\right) \\ & \ldots \\ & \bigcup \mathrm{WF}\left(F^{(|J_1|)}_{1\ \varphi}\right) \times \ldots \times \mathrm{WF}\left(F^{(|J_{n-1}|)}_{n-1\ \varphi}\right) \times \mathrm{supp}\left(F^{(|J_n|)}_{n\ \varphi}\right) \times \{\vec{0}\}^{|J_n|} \\ & \ldots \\ & \bigcup \mathrm{supp}\left(F^{(|J_1|)}_{1\ \varphi}\right) \times \{\vec{0}\}^{|J_1|} \times \ldots \times \mathrm{supp}\left(F^{(|J_{n-1}|)}_{n-1\ \varphi}\right) \times \{\vec{0}\}^{|J_{n-1}|} \times \mathrm{WF}\left(F^{(|J_n|)}_{n\ \varphi}\right). \end{aligned}$$

We clearly have that if an element of $\mathrm{WF}\left(F^{(|J_1|)}_1, \ldots, F^{(|J_n|)}_n\right)$ was contained in either $\overline{V}^k_{+,g}$ or $\overline{V}^k_{-,g}$ then at least one of the initial functional cannot be microcausal. □

Going through the same calculation for the proof of Proposition 5.4 we get the expression for the Peierls bracket of this composition:

$$\{\psi(F_1,\ldots,F_n),G\}_{\mathcal{L}}=\sum_{j=1}^{n}\left(\frac{\partial\psi}{\partial\mathsf{Re}(z)^j}(F_1,\ldots,F_n)\{\mathsf{Re}(F_j),G\}_{\mathcal{L}}+\frac{\partial\psi}{\partial\mathsf{Im}(z)^j}(F_1,\ldots,F_n)\{\mathsf{Im}(F_j),G\}_{\mathcal{L}}\right). \quad (60)$$

With the topology of Theorem 5.3 the space of microcausal functionals lacks sequential continuity. Consider as an example the simpler case where $B=M\times\mathbb{R}$, then choose $F:\varphi\in\mathcal{U}\mapsto\int_M\varphi(x)\omega$ for some smooth compactly supported m-form ω over M, also let $\{f_n\}$ be a sequence of smooth functions $:\mathbb{R}\to[0,1]$ supported in $[-2,2]$ and converging pointwise to the characteristic function of $[-1,1]$, $\chi_{[-1,1]}$, then sequences of derivatives of f_n all converge punctually to the zero function on $\mathbb{R}$. If we define $F_n(\varphi)=f_n\circ F(\varphi)$, then pointwise $F_n(\varphi)\to\chi_{[-1,1]}\circ F(\varphi)$ but

$$F^{(k)}_{n\ \varphi}[0](\psi_1,\ldots,\psi_k)=f_n^{(k)}(F(\varphi))\int_M\psi_1(x)\omega(x)\ldots\int_M\psi_k(x)\omega(x)$$

converges pointwise to the zero functional in the microcausal topology, however $\chi_{[-1,1]}\circ F(\cdot)$ is not even continuous, and definitely not microcausal.

Before the next result, let us recall some notions from [KM97]. A topological space (X,τ) is Lindelöf if given any open cover of X there is a countable open subcover, it is separable if it admits a countable dense subset, and it is second countable if it admits a countable basis for the topology.
Let X be a Hausdorff locally convex topological space, possibly infinite dimensional, and take $S\subset C(X,\mathbb{R})$, a subalgebra. We say that X is *S-normal* if $\forall A_0,A_1$ closed disjoint subsets of X there is some $f\in S$ such that $f|_{A_i}=i$, while we say it is *S-regular* if for any neighborhood U of a point x there exists a function $f\in S$ such that $f(x)=1$ and $\mathrm{supp}(f)\subset U$. A *$S$-partition of unity* is a family $\{\psi_j\}_{j\in J}$ of mappings $S\ni\psi_j:X\to\mathbb{R}$ with

1. $\psi_j(x)\geq 0$ for all $j\in J$ and $x\in X$;
2. the set $\{\mathrm{supp}(\psi_j):j\in J\}$ is a locally finite covering of X,
3. $\sum_{j\in J}\psi_j(x)=1$ for all $x\in X$.

When X admits such partition we say it is *S-paracompact.*

Proposition 5.5. *The following facts hold true:*

- (i) *Given any $\mathcal{U}\subset\Gamma^\infty(M\leftarrow B)$ CO-open and any $\varphi_0\in\mathcal{U}$ there is some $F\in\mathcal{F}_{\mu c}(B,\varphi_0,g)$ such that $F(\varphi_0)=1$, $0\leq F|_{\mathcal{U}}\leq 1$ and $F|_{\Gamma^\infty(M\leftarrow B)\setminus\mathcal{U}_{\varphi_0}}=0$, i.e. $\mathcal{U}$ is $\mathcal{F}_{\mu caus}(B,\varphi_0,g)$-regular.*
- (ii) *Any $\mathcal{U}\subset\Gamma^\infty(M\leftarrow B)$ CO-open admits locally finite partitions of unity belonging to $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$.*
- (iii) *Given any $\mathcal{U}\subset\Gamma^\infty(M\leftarrow B)$ CO-open, the algebra $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$ separates the points of $\mathcal{U}$, that is if $\varphi_1\neq\varphi_2$ there is a microcausal functional F that has $F(\varphi_1)\neq F(\varphi_2)$.*

Proof. To show (i) take the chart $(\mathcal{U}_{\varphi_0},u_{\varphi_0})$ and consider the open subset $\mathcal{U}\cap\mathcal{U}_{\varphi_0}$, fix some compact $K\subset M$ and some $\omega\in\Gamma^\infty_c(M\leftarrow\varphi^*VB'\otimes\Lambda_m(M))$ with $\mathrm{supp}(\omega)\subset K$, then we can define a functional

$$G_\omega:\mathcal{U}\cap\mathcal{U}_{\varphi_0}\ni\varphi\mapsto G_\omega(\varphi)=\int_M\omega(u_{\varphi_0}(\varphi)).$$

Denote now by G the functional with $G(\varphi)\doteq G_\omega\circ u^{-1}_{\varphi_0}(u_{\varphi_0}(\varphi))$. By construction $G(\varphi_0)=0$. Let now $\mathcal{W}=\{\varphi\in\mathcal{U}\cap\mathcal{U}_0:G(\varphi)<\epsilon^2\}$ for some constant ϵ, then if $\chi:\mathbb{R}\to\mathbb{R}$ is a smooth function supported in $[-1,1]$ with $0\leq\chi|_{[-1,1]}\leq 1$ and $\chi|_{[-1/2,1/2]}\equiv 1$, consider the new functional $F=\chi\circ(\frac{1}{\epsilon^2}G)$, since G is microlocal (and thus microcausal by Proposition 4.9) and χ is smooth, by Propositions 5.4 F is microcausal. Outside $\mathcal{W}$, F is identically zero so we can smoothly extend it to zero over the rest of $\mathcal{U}$ to a new functional which we denote always by F that has the required properties. We first show that (ii) holds for U_φ. Using the chart (U_φ,u_φ), we can identify U_φ with an open subset of $\Gamma^\infty_c(M\leftarrow\varphi^*VB)$. If we show that U_φ is Lindelöf and is $\mathcal{F}_{\mu c}(B,\varphi_0,g)$-regular, then we can conclude via Theorem 16.10 pp. 171 of [KM97]. $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$-regularity was point (i) while the Lindelöf property follows from Proposition 4.8 pp. 38 in [Mic80]. Now we observe that any $\mathcal{U}$ can be obtained as the disjoint union of subsets $\mathcal{V}_{\varphi_0}\doteq\{\psi\in\mathcal{U}:\mathrm{supp}_{\varphi_0}(\psi)\text{ is compact}\}$. Each of this is Lindelöf and metrizable by Proposition 42.3 pp. 440 in [KM97], so given the open cover $\{\mathcal{U}_\varphi\}_{\varphi\in\mathcal{V}_{\varphi_0}}$, we can extract a locally finite subcover where each elements admits a partition of unity and then construct a partition of unity for the whole $\mathcal{V}_{\varphi_0}$. The fact that $\mathcal{U}=\sqcup\mathcal{V}_{\varphi_0}$ implies that the final partition of unity is the union of all others. Finally for (iii) just take $\mathcal{U}_{\varphi_1}$, $\mathcal{U}_{\varphi_2}$ and F as in (i) constructed as follows: if $\varphi_2\in\mathcal{U}_1$ we choose $\epsilon<G(\varphi_2)$ for which $F(\varphi_1)\neq F(\varphi_2)$, if not then any $\epsilon>0$ does the job. $\square$

Definition 5.6. Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open and $\mathcal{L}$ a generalized microlocal Lagrangian. We define the on-shell ideal associated to $\mathcal{L}$ as the subspace $\mathcal{I}_{\mathcal{L}}(B,\mathcal{U},g) \subset \mathcal{F}_{\mu c}(B,\mathcal{U},g)$ whose microcausal functionals are of the form

$$F(\varphi) = \vec{X}_\varphi \left(E(\mathcal{L})_\varphi[0]\right) \tag{61}$$

with $X : \mathcal{U} \to T\mathcal{U} : \varphi \mapsto (\varphi, \vec{X}_\varphi)$ is a smooth vector field.

With our usual integral kernel notation we can also write (61) as

$$F(\varphi) = \int_M \vec{X}^i_\varphi(x) E(\mathcal{L})_\varphi[0]_i(x) d\mu_g(x). \tag{62}$$

We stress that functionals of the form (61) are those which can be seen as the derivation of the Euler-Lagrange derivative by kinematical vector fields over $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$.

Proposition 5.7. *$\mathcal{I}_{\mathcal{L}}(B,\mathcal{U},g)$ is a Poisson $*$-ideal of $\mathcal{F}_{\mu c}(B,\mathcal{U},g)$.*

Proof. Clearly is $F \in \mathcal{I}_{\mathcal{L}}(B,\mathcal{U},g)$ then also $\overline{F}$ is. If $G \in \mathcal{F}_{\mu c}(B,\mathcal{U},g)$, $G \cdot F(\varphi) = G(\varphi)X(\varphi)\left(E(\mathcal{L})_\varphi[0]\right)$ but then $X' = G \cdot X \in \mathfrak{X}(T\mathcal{U})$ as well, then $G \cdot F$ is in the ideal and is associated to the new vector field X'. Finally we have to show that if $F \in \mathcal{I}_{\mathcal{L}}(B,\mathcal{U},g)$ then also $\{F,G\}_{\mathcal{L}} \in \mathcal{I}_{\mathcal{L}}(B,\mathcal{U},g)$. Fix $\varphi \in \mathcal{U}$, $\vec{Y}_\varphi \in T_\varphi\mathcal{U}$; by the chain rule

$$F^{(1)}_\varphi[0](\vec{Y}_\varphi) = \int_M \left[\vec{X}^{(1)}_\varphi[0]^i(\vec{Y}_\varphi)\left(E(\mathcal{L})_\varphi[0]_i\right) + \vec{X}^i_\varphi\left(E^{(1)}(\mathcal{L})_\varphi[0]_i(\vec{Y}_\varphi)\right)\right] d\mu_g$$

and

$$\begin{aligned}
\{F,G\}_{\mathcal{L}}(\varphi) &= \left\langle F^{(1)}_\varphi[0], \mathcal{G}_\varphi G^{(1)}_\varphi[0]\right\rangle \\
&= \int_M \left[\vec{X}^{(1)}_\varphi[0]^i_j\left(\mathcal{G}^{jk}_\varphi g^{(1)}_\varphi[0]_k\right)\left(E(\mathcal{L})_\varphi[0]_i\right) + \vec{X}^i_\varphi\left(E^{(1)}(\mathcal{L})_\varphi[0]_{ij}\left(\mathcal{G}^{jk}_\varphi g^{(1)}_\varphi[0]_k\right)\right)\right] d\mu_g \\
&= \int_M \left\{\left[\vec{X}^{(1)}_\varphi[0]^i_j\left(\mathcal{G}^{jk}_\varphi g^{(1)}_\varphi[0]_k\right)\right]\left(E(\mathcal{L})_\varphi[0]_i\right)\right\} d\mu_g
\end{aligned}$$

where we used that $\mathcal{G}_\varphi$ associates to its argument a solution of the linearized equations. Defining $\varphi \mapsto \vec{Z}_\varphi = \vec{X}^{(1)}_\varphi[0]^i_j\left(\mathcal{G}^{jk}_\varphi g^{(1)}_\varphi[0]_k\right)\partial_i \in \Gamma^\infty_c(M \leftarrow \varphi^* VB)$ yields a smooth mapping (by smoothness of X, the functional G and the propagator $\mathcal{G}_\varphi$) defining the desired vector field. □

Definition 5.8. Let $\mathcal{U} \subset \Gamma^\infty(M \leftarrow B)$ be CO-open and $\mathcal{L}$ a generalized microlocal Lagrangian. We define the on-shell algebra on $\mathcal{U}$ associated to $\mathcal{L}$ as the quotient

$$\mathcal{F}_{\mathcal{L}}(B,\mathcal{U},g) \doteq \mathcal{F}_{\mu c}(B,\mathcal{U},g)/\mathcal{I}_{\mathcal{L}}(B,\mathcal{U},g). \tag{63}$$

This accounts for the algebra of observable once the condition $E(\mathcal{L})_\varphi[0] = 0$ has been imposed on $\mathcal{U}$.

6. Wave maps

Finally we introduce, as an example of physical theory, wave maps. The configuration bundle, is $C = M \times N$, where M is an m dimensional Lorentzian manifold and N an n dimensional manifold equipped with a Riemannian metric h. The space of sections is canonically isomorphic to $C^\infty(M,N)$, the latter possess a differentiable structure induced by the atlas $\left(\mathcal{U}_\varphi, u_\varphi, \Gamma^\infty_c(M \leftarrow \varphi^* TN)\right)$, where u_φ has the exact same form (5), with the only difference being that the sections are N-valued mappings, thus exp can be taken as the exponential function induced by a Riemannian metric h on N. The generalized Lagrangian for wave maps is

$$\mathcal{L}_{\mathrm{WM}}(f)(\varphi) = \int_M f(x)\frac{1}{2}\mathrm{Trace}(g^{-1} \circ (\varphi^* h))(x) d\mu_g(x); \tag{64}$$

obtained by integration of the standard geometric Lagrangian $\lambda = \frac{1}{2} g^{\mu\nu} h_{ij}(\varphi)\varphi^i_\mu \varphi^j_\nu d\mu_g$ smeared with a test function $f \in C^\infty_c(M)$. Computing the first functional derivative, as per (26), we get the associated E-L equations, which written in jets coordinates reads

$$h_{ij} g^{\mu\nu}\left(\varphi^i_{\mu\nu} + \{h\}^i_{kl}\varphi^k_\mu \varphi^l_\nu - \{g\}^\lambda_{\mu\nu}\varphi^i_\lambda\right) = 0, \tag{65}$$

where we denoted by $\{h\}$, $\{g\}$ the coefficients of the linear connection associated to h and g respectively. Computation of the second derivative of (64) yields

$$\begin{aligned}
&\delta^{(1)}E(\mathcal{L}_{WM})_\varphi[0] : \Gamma^\infty_c(M \leftarrow \varphi^*TN) \times \Gamma^\infty_c(M \leftarrow \varphi^*TN) \to \mathbb{R} \\
&\qquad (\vec{X}, \vec{Y}) \mapsto \int_M \frac{1}{2}\left[g^{\mu\nu}(x)h_{ij}(\varphi(x))\nabla_\mu X^i(x)\nabla_\nu Y^j(x) + A^\mu_{ij}(\varphi(x))\left(\nabla_\mu X^i(x)Y^j(x) + \nabla_\mu Y^i(x)X^j(x)\right)\right. \\
&\qquad\qquad\qquad \left. + B_{ij}(\varphi(x))X^i(x)Y^j(x)\right] d\mu_g(x).
\end{aligned} \tag{66}$$

Where we choose $f \equiv 1$ in a neighborhood of $\mathrm{supp}(\vec{X}) \cup \mathrm{supp}(\vec{X})$ as done before. One can show that the coefficients A^μ_{ij} always vanish and

$$\begin{aligned}
&\delta^{(1)}E(\mathcal{L}_{WM})_\varphi[0](\vec{X}, \vec{Y}) \\
&\quad = \int_M \frac{1}{2}\big(g^{\mu\nu}(x)h_{ij}(\varphi(x))\nabla_\mu X^i(x)\nabla_\nu Y^j(x) + R^k_{ilj}(\varphi(x))p^\alpha_k(\varphi(x))\varphi^l_\alpha(x)X^i(x)Y^j(x)\big)d\mu_g(x)
\end{aligned}$$

where R are the components of the Riemann tensor of the Riemannian metric h, and $p^\alpha_k \doteq \frac{\partial\lambda}{\partial y^k_\alpha}$ is the conjugate momenta of the Lagrangian λ. It is therefore evident that the induced differential operator D_φ can be expressed locally as

$$D_\varphi(\vec{X})(x) = \big(g^{\mu\nu}(x)h_{ij}(\varphi(x))\nabla_{\mu\nu}\vec{X}^i(x) + R^k_{ilj}(\varphi(x))p^\alpha_k(\varphi(x))\varphi^l_\alpha(x)X^i(x)\big)dy^j\big|_{\varphi(x)}. \tag{67}$$

Its principal symbol is clearly

$$\sigma_2(D_\varphi) = \frac{1}{2}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} \otimes id_{\varphi^*TN}.$$

Theorem 4.2 then ensures the existence of the advanced and retarded propagators for Wave Maps $\mathcal{G}^\pm_{WM}[\varphi]$. Their difference defines the causal propagator and consequently the Peierls bracket as in Definition 4.5. The results of sections 4 and 5 do apply to wave maps: it is therefore possible to obtain a $*$-Poisson algebra generated by microcausal functionals $\mathcal{F}_{\mu c}(M \times N, g)$ which enjoys all the properties collected throughout section 5.

7. Conclusions and Outlook

With the present paper we have partially explored the generalization of [BFR19] to space of configurations which are general fiber bundles. We remark that the main technical difficulties are the lack of a vector space structure for images of fields and the fact that while $C^\infty(M)$ is a Fréchet space, $\Gamma^\infty(M \leftarrow B)$ is not even a vector space. This forces us to use a manifold structure for $\Gamma^\infty(M \leftarrow B)$ and an appropriate calculus as well. For the manifold structure we choose locally convex spaces as modelling topological vector spaces. The notion of smooth mappings is however not unique, for instance, one could have used the convenient calculus of [KM97]; however, this calculus has the rather surprising property that smooth mappings need not be continuous (see [Glo05] for an example). This is rather annoying in view of the heavy usage of distributional spaces in sections 4 and 5, therefore we deemed more fitting Bastiani calculus (see [Bas64], [Mic38]).

As in the case of finite dimensional differentiable manifolds – where geometric properties can be equivalently be described locally but are independent from the local chart used – here to we tried to establish, where possible, Γ^∞-local independence for all notions introduced. It stands out though that the characterization of microlocality by Proposition 3.9 is, to the best of our knowledge, inherently Γ^∞-local. The argument supporting the Γ^∞-local character of Proposition 3.9 relies on the fact that a sufficient condition for Γ^∞-globality could be provided by a combination of *topological conditions* on the bundle and *regularity conditions* on the integral kernel of the first derivative of the functional. A noteworthy question would be which additional hypotheses, if any, could one add to Proposition 3.9 to make it a Γ^∞-global characterization.

Finally we mention that the definition of wave front set for distributional sections of vector bundles used, see 32, does leave some space for improvement, in particular one could use the refined notion of polarization wave front sets which appeared in [Den82] and attempt to re-derive all important result with this finer notion of singularity.

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