UNITARY, ANOMALOUS MASTER WARD IDENTITY AND ITS CONNECTIONS TO THE WESS-ZUMINO CONDITION, BV FORMALISM AND L_∞-ALGEBRAS

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ABSTRACT. The C*-algebraic construction of QFT by Buchholz and one of us relies on the causal structure of spacetime and a classical Lagrangian. In one of our previous papers we have introduced additional structure into this construction, namely an action of symmetries, which is related to fixing renormalisation conditions. This action characterizes anomalies and satisfies a cocycle condition which is summarized in the unitary anomalous Master Ward identity. Here (using perturbation theory) we show how this cocycle condition is related to the Wess-Zumino consistency relation and the consistency relation for the anomaly in the BV formalism, where the latter is the generalized Jacobi identity for the associated L_∞-algebra.

1. INTRODUCTION

One of the most interesting features of quantum physics is the fact that symmetries of the classical theory are, in general, not straightforwardly transferred to the corresponding quantum theory. Instead, often the symmetries are modified by *anomalies*. These satisfy the *Wess-Zumino consistency relations* [29], and the arising new structures have a crucial impact on the quantum theory, e.g. on the formulation of the standard model of particle physics.

In perturbative algebraic quantum field theory (pAQFT), the anomalies can be obtained in terms of the anomalous Master Ward Identity (AMWI) [4,12,14], and it was shown by Hollands [22] that in Yang-Mills theory these anomalies satisfy a consistency relation which allows one to apply the homological methods of the BRST formalism with antifields, where the key information about the theory is encoded in a certain differential. In [18,26] this result was generalized to arbitrary theories with local gauge symmetries by giving it the interpretation in terms of the infinite-dimensional rigorous version of the Batalin-Vilkovisky (BV) formalism. One of the crucial results of that work was to formulate the difference between classical symmetries and their quantized counterparts in terms of deformation of the classical BV differential to the quantum BV differential. This deformation is induced by the deformation of the pointwise product of the classical theory to the renormalized time-ordered product. In particular, a renormalized BV Laplacian was introduced, and its action on the BV algebra could be understood in terms of the anomaly [26]. Recently, Fröb [21] succeeded in proving that the arising algebraic structure is that of an L_∞-algebra. The Wess-Zumino consistency relation has also been applied recently in [27] in the treatment of global anomalies. Another insight concerns the difference between consistent anomalies, *i.e.* those that satisfy the Wess-Zumino conditions, and the so-called covariant anomalies [2]. We do not enter into this in our paper and refer the reader to the literature [3].

In a previous paper [8], we investigated the action of symmetries in the C*-algebraic construction of scalar quantum field theories proposed in [11]. In that construction the algebras are generated by S-matrices which describe local interactions within compact regions of spacetime. Subject to a causality condition and a unitary version of the Schwinger-Dyson equation, one obtains a net of C*-algebras satisfying the Haag-Kastler axioms, generalized to generic globally hyperbolic spacetimes according to the principles of locally covariant QFT [10]. Starting from the free Lagrangian

and admitting only linear interactions, one obtains the well known Weyl algebra of the free field. If one includes more general interactions, the arising algebra possesses automorphisms which act nontrivially only in a compact subregion. The existence of such internal symmetries violates the time slice axiom which states that observations in the neighborhood of some Cauchy surface determine all other observables, *i.e.* the algebra associated to this neighborhood is already the algebra of the whole spacetime.

Therefore, we introduced in [8] an additional axiom for these C*-algebras: the "unitary anomalous Master Ward Identity (UAMWI)." It characterizes how symmetries of the classical configuration space are modified in the quantum theory. The symmetries considered form a group $\mathcal{G}_c$ of transformations with compact support, generated by affine field redefinitions and point transformations. The modification consists in a transformation of the classical functional describing the local interaction. It leads to a map (the anomaly term) $\zeta : \mathcal{G}_c \to \mathcal{R}_c$ to a group $\mathcal{R}_c$ which is interpreted as the analogue to the Stückelberg-Petermann renormalization group which according to Stora's *Main Theorem of Renormalization* [15,25] governs the freedom of imposing renormalization conditions. And, most importantly for this paper, ζ satisfies a cocycle relation. We also showed that this cocycle ζ exists in the perturbative version of the model where it can be determined up to equivalence and yields the known anomalies.

Building on these results, in the present paper we explore further the cocycle condition, focusing on perturbation theory. Since the non-triviality of this cocycle is related to the existence of anomalies, it is reasonable to expect that it should be related to the Wess-Zumino consistency condition. The latter has been originally derived in the context of the effective action and it reflects to the way in which this action transforms under infinitesimal gauge symmetries. In Section 3 we review that original derivation, following essentially [1]. Although it is clear that the Wess-Zumino consistency condition has to be related to the action of the Lie algebra of the group of symmetries of the theory, the precise statement of this fact in the framework of [8,11] has not been known. While addressing this question, the present work also makes connections with another statement of the Wess-Zumino consistency condition, namely the one present in the BV formalism.

Concretely, we show that, considering the infinitesimal symmetry transformations, the cocycle ζ induces a corresponding map $\Delta : \text{Lie}\,\mathcal{G}_c \to \text{Lie}\,\mathcal{R}_c$ which is a Lie algebraic cocycle, and that this cocycle is the anomaly map appearing in the AMWI (Theorem 10.3 in [8] and Theorem 5.1 in this paper). This provides a link between the notions of anomalies used in perturbation theory [4] and anomalies in the non-perturbative formulation of [8]. In Section 4, in Theorem 4.1, we give another derivation of the cocycle relation for Δ: we show that the anomaly Δ of the AMWI satisfies a consistency condition, which is precisely the cocycle relation for Δ, and which we call the *extended Wess-Zumino consistency condition*, as it reduces to the standard Wess-Zumino condition for quadratic interactions.

Finally, we discuss the relation to the BV formalism. In [18], two of us have shown that the anomaly in the AMWI is in fact related to the *renormalised BV Laplacian*, so it is natural to expect that the algebraic properties of the BV Laplacian would be reflected also in the cocycle condition. This is indeed the case, as we prove in Section 6 that the extended cocycle condition for Δ follows directly from the nilpotency of the BV operator, when applied to those infinitesimal symmetries which arise from affine field redefinitions $\mathfrak{g} \in \mathcal{G}_c$ (Prop. 6.6).

2. The framework

2.1. **Perturbative algebraic quantum field theory (pAQFT).** We use the same setting for pAQFT as in [8, Sect. 10 and App. C]; for the convenience of the reader we repeat here in a somewhat sketchy way the notations, definitions and results being relevant for this paper.

We consider an n-component real scalar field Φ on a globally hyperbolic curved space-time M of dimension larger than 2. The classical configuration space $\mathscr{E}(M, \mathbb{R}^n)$ is the space of smooth

functions on M with values in $\mathbb{R}^n$. The basic field $\Phi(x)$ is the evaluation functional

$$\Phi(x) \,:\, \mathscr{E}(M, \mathbb{R}^n) \to \mathbb{R}^n \,;\, \Phi(x)[\phi] = \phi(x) \,. \tag{2.1}$$

Observables are elements of the space $\mathscr{F}(M)$ of functionals $F : \mathscr{E}(M, \mathbb{R}^n) \to \mathbb{C}$ which are polynomial in ϕ and have the form

$$F[\phi] = \sum_{k=0}^{m} \langle f_k, \phi^{\otimes k} \rangle \tag{2.2}$$

with compactly supported distributional densities f_k on M^k satisfying suitable conditions on their wave front sets [6,9]. The latter ensures the existence of the star product of the free theory (which is given in terms of the free Lagrangian L, see below) as a map $\star : \mathscr{F}(M) \times \mathscr{F}(M) \to \mathscr{F}(M)$. This star product is an $\hbar$-dependent deformation of the (commutative) pointwise product: $F \cdot G[\phi] \doteq F[\phi]G[\phi]$ for $F, G \in \mathscr{F}$, $\phi \in \mathscr{E}(M, \mathbb{R}^n)$, see [13] or [16, Chap. 2]. The (functional) support of a functional F as above is the smallest closed set $K \subset M$ such that $\operatorname{supp} f_k \subset K^k$ for all k (where $\operatorname{supp} f_0 = \emptyset$ is understood).

The subspace of *local* functionals $F \in \mathscr{F}_{\mathrm{loc}}(M)$ is defined by the additional conditions that F is $\mathbb{R}$-*valued* and of the form $F[\phi] = \int \hat{F}(x, j_x(\phi))$, with a smooth density-valued function $\hat{F}$ on the jet space of $\mathscr{E}(M, \mathbb{R}^n)$ with compact support in x.

The Lagrangian L is the standard Lagrangian of the free theory where we use the canonical metric on $\mathbb{R}^n$; $L(x)$ is a density with values in the local functionals and we write $L(f) \doteq \int_M L(x)f(x) \in \mathscr{F}_{\mathrm{loc}}(M)$ for $f \in \mathscr{D}(M, \mathbb{R})$.

To construct the time ordered product we use an off-shell version of the Epstein-Glaser method[1] [17], generalized to globally hyperbolic space times [9,23,24]. We furthermore use the fact that the k-fold pointwise product of local functionals which vanish at the zero configuration is injective and thus isomorphic to its image, the k-local functionals[2] $F \in \mathscr{F}_{k\,\mathrm{loc}}(M)$. Identifying the 1-fold product with the identity and the 0-fold product with the map $\mathbb{R} \ni c \mapsto F_c$ with the constant functional $F_c[\phi] = c$, we can describe the time ordered product as a linear map

$$T : \mathscr{F}_{\bullet\,\mathrm{loc}}(M) \to \mathscr{F}(M) \tag{2.3}$$

where $\mathscr{F}_{\bullet\,\mathrm{loc}}(M)$, the space of multilocal functionals, is the direct sum of the spaces $\mathscr{F}_{k\,\mathrm{loc}}(M)$ of k-local functionals, $k \in \mathbb{N}_0$ [18]. We can then equip the space $T\mathscr{F}_{\bullet\,\mathrm{loc}}(M)$ with the commutative and associative product

$$F \cdot_T G \doteq T((T^{-1}F) \cdot (T^{-1}G)) \,. \tag{2.4}$$

On local functionals T is the identity. In the sense of formal power series we can then characterize T by its action on exponentials of local functionals $S(F) = Te^{iF} \equiv e_{\cdot_T}^{iF}$, the *formal S-matrices*. They are unitaries with respect to the $\star$-product and have to satisfy the condition of *causal factorization*

$$S(F + G) = S(F) \star S(G) \quad \text{if } \operatorname{supp} F \cap J_-(\operatorname{supp} G) = \emptyset \tag{2.5}$$

where J_- denotes the past of a space-time region and $F, G \in \mathscr{F}_{\mathrm{loc}}(M)$. The time-ordered product is further restricted by renormalization conditions: in this paper we also require field independence

$$\frac{\delta}{\delta \phi} T(F) = T\left(\frac{\delta}{\delta \phi} F\right), \quad F \in \mathscr{F}_{\bullet\,\mathrm{loc}}(M), \tag{2.6}$$

and the off-shell field equation

$$T(F \cdot \langle \Phi, f \rangle) = T(\langle F', E^{\mathrm{F}} f \rangle) + T(F) \cdot \langle \Phi, f \rangle \,, \quad f \in \mathscr{D}(M, \mathbb{R}^n), \quad F \in \mathscr{F}_{\bullet\,\mathrm{loc}}(M), \tag{2.7}$$

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[1]Epstein and Glaser consider Fock space operators of the form $\sum_k \langle f_k, :\varphi^{\otimes k}: \rangle$ with the normal ordered products of the free field φ. This corresponds to a restriction of functionals to the space of solutions of the free field equation (on shell formalism).

[2]Note that $\mathscr{F}_{\mathrm{loc}}(M) = \mathscr{F}_{0\,\mathrm{loc}}(M) + \mathscr{F}_{1\,\mathrm{loc}}(M)$.

where E^{F} is the Feynman propagator and F' is the first derivative of F. Note that the non-uniqueness of the Feynman propagator on curved spacetimes leads to an ambiguity of the time-ordered product which however is irrelevant since it is absorbed by a corresponding freedom in the association of observables to functionals by a normal ordering procedure, see [19, 20] for details. Time ordered products satisfying these axioms exist, and, according to Stora's Main Theorem of Renormalization [15, 25], any two formal S-matrices S and $\hat{S}$ are related by

$$\hat{S} = S \circ Z \tag{2.8}$$

where Z is a formal power series

$$Z(F) = \sum_{n=1}^{\infty} \frac{1}{n!} Z_n(F^n) \tag{2.9}$$

of linear maps $Z_n : \mathscr{F}_{n\,\mathrm{loc}}(M) \to \mathscr{F}_{\mathrm{loc}}(M)$. The *Stückelberg-Petermann renormalization group* $\mathscr{R}_0$ is defined to be the set of maps $Z \equiv (Z_n)_{n\in\mathbb{N}}$ appearing in (2.8), and one proves that this set is indeed a group [15]. For a direct definition of $\mathscr{R}_0$ see [6, 15] or [16, Chap. 3.6].

We immediately see that $Z_1 = \mathrm{id}$. To include also possible changes of the Feynman propagator which is unavoidable in a generally covariant formalism [24], we generalize the definition of $\mathscr{R}_0$ by admitting nontrivial, but still invertible Z_1 which describe the change of the normal ordering and thus the action of the time ordering operator $\hat{T}$ on 1-local functionals. For convenience, we continue to use a time ordering operator T which is the identity on local functionals and obtain the more general time orderings by composition with the renormalization group map Z as in equation (2.8).

2.2. **Unitary anomalous master Ward identity.** In pAQFT, the unitary anomalous master Ward identity (UAMWI) describes the behaviour of the time-ordered product under the group $\mathcal{G}_c(M)$ of compactly supported automorphisms of the affine bundle $M \times \mathbb{R}^n$. This group is generated by the following transformations $g : \mathscr{E}(M, \mathbb{R}^n) \to \mathscr{E}(M, \mathbb{R}^n)$:

- *Point transformations*, i.e. smooth and compactly supported diffeomorphisms $\rho : M \to M$ inducing the transformation $g_\rho : \phi \mapsto g_\rho(\phi) \doteq \phi \circ \rho$.
- *Affine field redefinitions* $g_{(A,\psi)}$ with $A \in \mathscr{D}(M, \mathrm{GL}(n, \mathbb{R}))$ and $\psi \in \mathscr{D}(M, \mathbb{R}^n)$ which act on configurations by

$$g_{(A,\psi)}(\phi)(x) \doteq \phi(x)A(x) + \psi(x) . \tag{2.10}$$

  where $\phi(x)$ and $\psi(x)$ are considered as row vectors.

The action of $\mathcal{G}_c(M)$ on a functional $F \in \mathscr{F}_{\mathrm{loc}}(M)$ is defined by

$$g_* F[\phi] \doteq F[g(\phi)] \tag{2.11}$$

and the free Lagrangian L is transformed by

$$((g_\rho)_* L)(f) \doteq (g_\rho)_*(L(f \circ \rho)) , \quad ((g_{(A,\psi)})_* L)(f) \doteq (g_{(A,\psi)})_*(L(f)) \tag{2.12}$$

with $f \in \mathscr{D}(M, \mathbb{R})$. Note that $(gh)_* = g_* h_*$ for $g, h \in \mathcal{G}_c(M)$, that is, $\mathcal{G}_c(M) \ni g \mapsto g_*$ is a representation of $\mathcal{G}_c(M)$ by maps on $\mathscr{F}_{\mathrm{loc}}(M)$.

The group $\mathcal{G}_c(M)$ acts on the full Lagrangian, and hence on the interaction by an L-dependent action on $\mathscr{F}_{\mathrm{loc}}(M)$

$$(g, F) \mapsto g_L F \doteq \delta_g L + g_* F \quad \text{where} \quad \delta_g L \doteq g_* L(f) - L(f) \tag{2.13}$$

with $f \in \mathscr{D}(M, \mathbb{R})$ such that $f|_{\mathrm{supp}\,g} = 1$. Obviously $e_L = \mathrm{id}_{\mathscr{F}_{\mathrm{loc}}(M,L)}$ for the unit $e \in \mathcal{G}_c(M)$, and one verifies that $(gh)_L = g_L \circ h_L$.

The unitary anomalous master Ward identity (UAMWI) relates the transformations induced by the action $g \to g_L$ to renormalization group transformations $\zeta_g \in \mathscr{R}_0$ with $\mathrm{supp}\,\zeta_g = \mathrm{supp}\,g$. Here the support of $Z \in \mathscr{R}_0$ is the smallest closed subset N of M such that $Z(F + G) = F + Z(G)$ for all $F, G \in \mathscr{F}_{\mathrm{loc}}(M)$ with $\mathrm{supp}\,F \cap N = \emptyset$. The subgroup $\mathscr{R}_c$ of $\mathscr{R}_0$ of renormalization group maps Z with *compact support* was discussed in [8, Appendix C].

We will now discuss the UAMWI in pAQFT. To allow for an off-shell description, we introduce "sources." Let $q \in \mathscr{E}_{\mathrm{dens}}(M, \mathbb{R}^n)$ be a smooth density and we define $L_q \doteq L - \langle \Phi, q \rangle$. In pAQFT, the UAMWI states that there exists a map (called "anomaly map")

$$\zeta : \mathcal{G}_c(M) \to \mathscr{R}_c, \quad \text{satisfying} \quad \zeta_e = \mathrm{id}_{\mathscr{F}_{\mathrm{loc}}(M)}, \quad \mathrm{supp}\, \zeta_g \subset \mathrm{supp}\, g \tag{2.14}$$

and the *cocycle relation*

$$\zeta_{gh} = \zeta_h\, h_L^{-1} \zeta_g\, h_L\, , \quad g, h \in \mathcal{G}_c(M)\, , \tag{2.15}$$

such that for every smooth density $q \in \mathscr{E}_{\mathrm{dens}}(M, \mathbb{R}^n)$

$$S \circ g_{L_q}(F)[\phi] = S \circ \zeta_g(F)[\phi]\, , \text{ for } \phi \text{ solving } \frac{\delta L}{\delta \phi}[\phi] = q\, , \tag{2.16}$$

with $g \in \mathcal{G}_c(M),\ F \in \mathscr{F}_{\mathrm{loc}}(M)$ arbitrary. As shown in Theorem 10.3 in [8], the UAMWI follows[3] from the anomalous master Ward identity (AMWI) [4] (recalled below in (2.24) or (2.26)), which is its infinitesimal version, formulated in terms of the respective Lie algebras.

The Lie algebra $\mathrm{Lie}\,\mathscr{R}_c$ is defined as follows (compare [8, Appendix C]): it is the space of formal power series $z(F) = \sum_{n=1}^{\infty} \frac{1}{n!} z_n(F^n)$, with linear maps $z_n : \mathscr{F}_{n\,\mathrm{loc}}(M) \to \mathscr{F}_{\mathrm{loc}}(M)$, with the properties

(P1) $\mathrm{id} + \lambda z_1$ is invertible for λ sufficiently small,
(P2) $z(F + G) = z(F) + z(G)$ for $\mathrm{supp}\, F \cap \mathrm{supp}\, G = \emptyset$, $F, G \in \mathscr{F}_{\mathrm{loc}}(M)$,
(P3) $z(F + \langle \Phi, \psi \rangle) = z(F)$ for $\psi \in \mathscr{D}(M, \mathbb{R}^n)$,
(P4) $\frac{\delta}{\delta \phi} z(F) = \langle z'(F), (\frac{\delta}{\delta \phi} F) \rangle\, ,$
(P5) the support of z is compact, where $\mathrm{supp}\, z$ is the smallest closed subset N of M such that $z(F + G) = z(G)$ for all $F, G \in \mathscr{F}_{\mathrm{loc}}(M)$ with $\mathrm{supp}\, F \cap N = \emptyset$.

The action of $\mathcal{G}_c(M)$ on the configuration space (considered as an affine space) induces an action of the Lie algebra $\mathrm{Lie}\,\mathcal{G}_c(M)$ with values in the associated vector space,

$$\mathscr{E}(M, \mathbb{R}^n) \times \mathrm{Lie}\,\mathcal{G}_c(M) \ni (\phi, X) \mapsto \phi X\, . \tag{2.17}$$

To determine the Lie bracket, it is convenient to describe the Lie algebra $\mathrm{Lie}\,\mathcal{G}_c(M)$ in a faithful representation of the group. Since $\mathcal{G}_c(M)$ acts from the right on field configurations, we write it in terms of a matrix multiplication from the right on the space $\mathscr{E}(M, \mathbb{R}^n) \oplus \mathbb{R}$ in the form

$$g : (\phi, c) \mapsto (\phi, c) \begin{pmatrix} A & 0 \\ \psi & 1 \end{pmatrix} \circ \begin{pmatrix} \rho & 0 \\ 0 & \mathrm{id} \end{pmatrix}\, , \tag{2.18}$$

with $g = (A, \psi, \rho)$, from which we get

$$X : (\phi, c) \mapsto (\phi, c) \begin{pmatrix} a + \overleftarrow{\partial}_\mu v^\mu & 0 \\ p & 0 \end{pmatrix} = (\phi a + v^\mu \partial_\mu \phi + cp, 0) \doteq (\phi X, 0) \tag{2.19}$$

with the Lie algebra element $X = (a, p, v)$, $a \in \mathscr{D}(M, \mathrm{gl}(n, \mathbb{R}))$, $p \in \mathscr{D}(M, \mathbb{R}^n)$ and a smooth vector field v with compact support. The Lie bracket

$$[(a, p, v), (b, q, w)] = ([a, b] + w^\nu \partial_\nu a - v^\nu \partial_\nu b, pb - qa + w^\nu \partial_\nu p - v^\nu \partial_\nu q, w^\nu \partial_\nu v - v^\nu \partial_\nu w) \tag{2.20}$$

can directly be obtained from the matrix representation above. Note the unusual sign of the Lie bracket of vector fields due to the action of derivatives to the functions on the left, indicated by the upper left arrow.

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[3]In [8] only the case $q = 0$ was treated. The generalization to arbitrary densities q relies on the fact that ζ does not change under adding a source term $-\langle \Phi, q \rangle$ to the Lagrangian, see Theorem A.1 in appendix A. In particular, the proof of that Theorem explicitly shows that the cocycle relation (2.15) is a necessary condition for the UAMWI (2.16).

The action of $\mathrm{Lie}\,\mathcal{G}_c(M)$ on field configurations yields a representation $X \mapsto \partial_X$ on the space of local functionals with

$$\partial_X F[\phi] = \langle F'[\phi], \phi X \rangle \equiv \int \frac{\delta F}{\delta \phi_a(x)}[\phi](\phi X)_a(x) \,, \tag{2.21}$$

where we use the usual summation conventions over the components of ϕ, $a = 1, \ldots, n$; and the functional derivative is naturally identified with a density. In particular we have

$$[\partial_X, \partial_Y] = \partial_{[X,Y]} \,. \tag{2.22}$$

For the Lagrangian, we set

$$\partial_X L \doteq \partial_X L(f) \quad \text{with } f \in \mathscr{D}(M, \mathbb{R}) \text{ satisfying } f|_{\mathrm{supp}\,X} = 1. \tag{2.23}$$

To derive the AMWI from the UAMWI let $X \in \mathrm{Lie}\,\mathcal{G}_c(M)$ be the tangent vector at $\lambda = 0$ of a smooth curve $\lambda \mapsto g^\lambda \in \mathcal{G}_c(M)$ with $g^0 = e$. Starting with the UAMWI (2.16) we substitute g^λ for g and apply $\frac{d}{d\lambda}|_{\lambda=0}$; this yields

$$T\Big(e^{iF} \cdot \big(\partial_X F + \partial_X L_q - \Delta X(F)\big)\Big)[\phi] = 0 \,, \text{ for } \phi \text{ solving } \frac{\delta L}{\delta \phi}[\phi] = q \,, \tag{2.24}$$

with $X \in \mathrm{Lie}\,\mathcal{G}_c(M), F \in \mathscr{F}_{\mathrm{loc}}(M)$ and where

$$\Delta \,:\, \mathrm{Lie}\,\mathcal{G}_c(M) \ni X \mapsto \Delta X \doteq \frac{d}{d\lambda}\Big|_{\lambda=0} \zeta_{g^\lambda} \in \mathrm{Lie}\,\mathscr{R}_c \,. \tag{2.25}$$

Indeed, (2.24) agrees with the AMWI, thus $\Delta X(F)$ coincides with the uniquely determined anomaly in the AMWI (see [4, Thm. 7], [5, Thm. 5.2] and [16, Chap. 4.3]). The AMWI (2.24) may also be written in the equivalent form

$$e_T^{iF} \cdot_T \big(\partial_X F + \partial_X L - \Delta X(F)\big) = \int \big(e_T^{iF} \cdot_T (\partial_X \Phi(x))_a\big) \frac{\delta L(f)}{\delta \phi_a(x)} \,, \ f \equiv 1 \text{ on supp } X \,, \tag{2.26}$$

where $\delta L(f)/\delta \phi_a(x)$ is understood as a density. We observe that the map $X \mapsto \Delta X$ is linear and that $\mathrm{supp}\,\zeta_g \subset \mathrm{supp}\,g$ implies $\mathrm{supp}\,\Delta X \subset \mathrm{supp}\,X$. Moreover, there is a common locality of $\Delta X(F)$ in X and F derived in [4, Thm. 7], see also [16, Thm. 4.3.1]:

Lemma 2.1. *The anomaly map Δ of the AMWI satisfies*

$$\mathrm{supp}\,\Delta X(F) \subset \mathrm{supp}\,F \cap \mathrm{supp}\,X. \tag{2.27}$$

and

$$\Delta X(F) = 0 \quad \text{if} \quad \mathrm{supp}\,F \cap \mathrm{supp}\,X = \emptyset \,. \tag{2.28}$$

3. Review of the Wess-Zumino consistency relations

Following [1], we concentrate here on the subgroup $\mathcal{G}_o \subset \mathcal{G}_c$ of orthogonal field redefinitions $\phi \mapsto g\phi$ where $g : M \mapsto \mathrm{SO}(n, \mathbb{R})$ is smooth and compactly supported and ϕ is written as a column vector. An orthogonal field redefinition may be interpreted as a gauge transformation. It transforms the trivial connection on the vector bundle $M \times \mathbb{R}^n$ to an equivalent one which may be considered as an external gauge field A which is a pure gauge, *i.e.* $A = g^{-1}dg$ for some $g \in \mathcal{G}_o$. We consider Lagrangians L_A,

$$L_A(\phi) = \frac{1}{2}\langle (d + A)\phi, (d + A)\phi \rangle \,, \tag{3.1}$$

which depend on a compactly supported external gauge field A, considered as a $\mathfrak{so}(n, \mathbb{R})$-valued 1-form, in symbols $A \in \Omega_c^1(M, \mathfrak{so}(n))$. The bracket here combines the spacetime metric on 1-forms together with the canonical inner product on $\mathbb{R}^n$. We have

$$(g_* L_A)(\phi) = L_A(g^{-1}\phi) = L_{A^g}(\phi) \,, \tag{3.2}$$

where $A^g = g(dg^{-1}) + gAg^{-1}$ is the gauge transformed gauge field. Let $V(A) \doteq \int (L_A - L)$ denote the interaction induced by A. Then

$$V(A^g) = g_L V(A), \tag{3.3}$$

where g_L refers to the action of $\mathcal{G}_o$ on ϕ defined in (2.13). One now considers the effective action, *i.e.* the Legendre transform of the generating functional of connected Green's functions,

$$\Gamma(A, \varphi) \doteq \langle \varphi, J \rangle - i \log \big(S(V(A) + \langle \Phi, J \rangle)[\phi = 0] \big) , \quad \varphi \in \mathscr{D}(M, \mathbb{R}^n) , \tag{3.4}$$

where $J(\varphi, A)$ is the solution of

$$\varphi = -i \frac{\delta}{\delta j} \log \big(S(V(A) + \langle \Phi, j \rangle)[\phi = 0] \big) \big|_{j=J} . \tag{3.5}$$

Since L_A is a quadratic functional of ϕ, we have the explicit solution

$$J = \Box_A \varphi \tag{3.6}$$

with the d'Alembertian $\Box_A$ for an external gauge field A. Thus

$$\Gamma(A, \varphi) = \int L_A(\varphi) - i \log \big(S(V(A) + \langle \Phi, \Box_A \varphi \rangle)[\phi = 0] \big) . \tag{3.7}$$

In the absence of anomalies Γ should be gauge invariant. The action of the gauge group on gauge fields $A \mapsto A^g$ and matter fields $\varphi \mapsto g\varphi$ induces a corresponding representation $X \mapsto \partial_X^{A,\varphi}$ of the Lie algebra, acting by derivations on functions K of these fields,

$$\partial_X^{A,\varphi} K(A, \varphi) \doteq \frac{d}{d\lambda}\big|_{\lambda=0} K(A^{g_\lambda}, g_\lambda \varphi) \tag{3.8}$$

with $g_\lambda = \exp(-\lambda X)$; in particular $[\partial_X^{A,\varphi}, \partial_Y^{A,\varphi}] = \partial_{[X,Y]}^{A,\varphi}$ holds. One defines the anomaly by

$$G(X, A) \doteq \partial_X^{A,\varphi} \Gamma(A, \varphi) . \tag{3.9}$$

Even though Γ is nonlocal and depends on φ, the anomaly is a local functional of A and independent of φ. These two statements are well known from the literature (see e.g. [28]), but for completeness we give independent proofs below. An immediate consequence of the definition (3.9) is the *Wess-Zumino consistency relation*

$$\partial_X^A G(Y, A) - \partial_Y^A G(X, A) = G([X, Y], A) , \tag{3.10}$$

where we write ∂_X^A instead of $\partial_X^{A,\varphi}$ when acting on a functional not depending on φ. The consistency relation is a nontrivial restriction on the structure of anomalies, although it is an obvious consequence of the fact that anomalies are defined directly through a Lie elgebra action.

Next we show how this consistency relation can be derived directly from the UAMWI. For $g \in \mathcal{G}_o$, we have that

$$V(A^g) + \langle \Phi, \Box_{A^g}(g\varphi) \rangle = g_L \big(V(A) + \langle \Phi, \Box_A \varphi \rangle \big) . \tag{3.11}$$

Using the UAMWI, we get

$$S\big(V(A^g) + \langle \Phi, \Box_{A^g}(g\varphi) \rangle \big)\big|_{\phi=0} = S\big(\zeta_g (V(A) + \langle \Phi, \Box_A \varphi \rangle) \big)\big|_{\phi=0} . \tag{3.12}$$

Using parts (i) and (iii) of Prop. 4.14 in [8], we obtain

$$\zeta_g \big(V(A) + \langle \Phi, \Box_A \varphi \rangle \big) = \zeta_g \big(V(A) \big) + \langle \Phi, \Box_A \varphi \rangle = V(A) + \langle \Phi, \Box_A \varphi \rangle + \mathfrak{G}(g, A), \tag{3.13}$$

with a functional $\mathfrak{G}$ which depends on g and A but does neither depend on the field φ nor on the field configuration ϕ. Moreover $\mathfrak{G}$ is local in the sense that for $x \neq y$

$$\frac{\delta^2 \mathfrak{G}}{\delta A(x) \delta A(y)} = 0 \quad , \quad \frac{\delta^2 \mathfrak{G}}{\delta g(x) \delta A(y)} = 0 \quad \text{and} \quad \frac{\delta^2 \mathfrak{G}}{\delta g(x) \delta g(y)} = 0 . \tag{3.14}$$

This follows from $\mathfrak{G}(g, A) = \zeta_g(V(A)) - V(A)$ (3.13) and the following proposition:

Proposition 3.1. *Let* $\omega, \omega_1, \omega_2 \in \Omega_c^1(M, \mathfrak{so}(n))$, $g, g_1, g_2, h \in \mathcal{G}_0$ *such that* $\operatorname{supp} \omega_1 \cap \operatorname{supp} \omega_2 = \emptyset$, $\operatorname{supp} \omega \cap \operatorname{supp} g = \emptyset$ *and* $\operatorname{supp} g_1 \cap \operatorname{supp} g_2 = \emptyset.$ *Then*

    (1) $\zeta_h\big(V(A + \omega_1 + \omega_2)\big) = \zeta_h\big(V(A + \omega_1)\big) - \zeta_h\big(V(A)\big) + \zeta_h\big(V(A + \omega_2)\big)$,

    (2) $\zeta_{gh}\big(V(A + \omega)\big) = \zeta_h\big(V(A + \omega)\big) - \zeta_h\big(V(A)\big) + \zeta_{gh}\big(V(A)\big)$,

    (3) $\zeta_{g_1 g_2 h}\big(V(A)\big) = \zeta_{g_1 h}\big(V(A)\big) - \zeta_h\big(V(A)\big) + \zeta_{g_2 h}\big(V(A)\big)$.

Proof.   (1) From $V(A) = \int(L_A - L))$ we see that V is a local functional of A, hence

$$V(A + \omega_1 + \omega_2) = V(A + \omega_1) - V(A) + V(A + \omega_2) \tag{3.15}$$

and $\operatorname{supp}(V(A + \omega_i) - V(A)) \subset \operatorname{supp} \omega_i$, $i = 1, 2$. Since ζ_h satisfies the additivity relation

$$\zeta_h(F + G + H) = \zeta_h(F + G) - \zeta_h(G) + \zeta_h(G + H) \tag{3.16}$$

for $F, G, H \in \mathscr{F}_{\mathrm{loc}}(M)$ with $\operatorname{supp} F \cap \operatorname{supp} H = \emptyset$, we get

$$\zeta_h\big(V(A + \omega_1 + \omega_2)\big) = \zeta_h\big((V(A + \omega_1) - V(A)) + V(A) + (V(A + \omega_2) - V(A))\big)$$
$$= \zeta_h\big(V(A + \omega_1)\big) - \zeta_h\big(V(A)\big) + \zeta_h\big(V(A + \omega_2)\big) . \tag{3.17}$$

    (2) From the cocycle relation we have $\zeta_{gh} = \zeta_h \zeta_g^h$ with $\zeta_g^h \doteq h_L^{-1} \zeta_g h_L \in \mathscr{R}_c$ [8, Lemma 5.4]. In the first step we show that $\operatorname{supp} \zeta_g^h \subset \operatorname{supp} \zeta_g$.

Let $F, G \in \mathscr{F}_{\mathrm{loc}}(M)$ and $\operatorname{supp} G \cap \operatorname{supp} g = \emptyset$. With $\operatorname{supp} h_* G = \operatorname{supp} G$ we get

$$\zeta_g^h(F + G) = h_L^{-1} \zeta_g(h_L F + h_* G)$$
$$= h_L^{-1}\big(\zeta_g(h_L F) + h_* G\big)$$
$$= \zeta_g^h(F) + G . \tag{3.18}$$

Then $\operatorname{supp}\big(\zeta_g^h(V(A)) - V(A)\big) \subset \operatorname{supp} \zeta_g^h \subset \operatorname{supp} g$ (see [8, Prop. 4.14(ii)]) and we find that

$$\zeta_{gh}\big(V(A + \omega)\big) = \zeta_h \zeta_g^h\big((V(A + \omega) - V(A)) + V(A)\big)$$
$$= \zeta_h\big(\zeta_g^h(V(A)) + V(A + \omega) - V(A)\big)$$
$$= \zeta_h\big((\zeta_g^h(V(A)) - V(A)) + V(A) + (V(A + \omega) - V(A))\big)$$
$$= \zeta_h \zeta_g^h\big(V(A)\big) - \zeta_h\big(V(A)\big) + \zeta_h\big(V(A + \omega)\big) . \tag{3.19}$$

    (3) In the first step we show that for $g, h \in \mathcal{G}_o$ with disjoint supports (i.e. $\operatorname{supp} g \cap \operatorname{supp} h = \emptyset$) the following relation holds for $F \in \mathscr{F}_{\mathrm{loc}}(M)$:

$$\zeta_g^h(F) = h_L^{-1} \zeta_g h_L(F)$$
$$= h_L^{-1} \zeta_g\big((h_L F - F) + F\big)$$
$$= h_L^{-1}\big(h_L F - F + \zeta_g(F)\big)$$
$$= F - h_L^{-1} F + h_L^{-1} \zeta_g(F)$$
$$= F - h_L^{-1} F + h_L^{-1}\big((\zeta_g(F) - F) + F\big)$$
$$= F - h_L^{-1} F + \zeta_g(F) - F + h_L^{-1} F$$
$$= \zeta_g(F) . \tag{3.20}$$

Here we used that $\mathrm{supp}\,(\hbar_L F - F) \subset \mathrm{supp}\,\hbar$ and $\mathrm{supp}\,(\zeta_g(F) - F) \subset \mathrm{supp}\,g$. We now compute

$$
\begin{aligned}
\zeta_{g_1 g_2 \hbar}(F) &= \zeta_\hbar \hbar_L^{-1} \zeta_{g_2} \zeta_{g_1}^{g_2} \hbar_L(F) \\
&= \zeta_\hbar \hbar_L^{-1} \zeta_{g_2} \zeta_{g_1} \hbar_L(F) \\
&= \zeta_\hbar \hbar_L^{-1} \zeta_{g_2} \big((\zeta_{g_1} \hbar_L(F) - \hbar_L(F)) + \hbar_L(F) \big) \\
&= \zeta_\hbar \hbar_L^{-1} \big((\zeta_{g_1} \hbar_L(F) - \hbar_L(F)) + \zeta_{g_2} \hbar_L(F) \big) \\
&= \zeta_\hbar \big(\zeta_{g_1}^{\hbar}(F) - F + \zeta_{g_2}^{\hbar}(F) \big) \\
&= \zeta_\hbar \big((\zeta_{g_1}^{\hbar}(F) - F) + F + (\zeta_{g_2}^{\hbar}(F) - F) \big) \\
&= \zeta_\hbar \zeta_{g_1}^{\hbar}(F) - \zeta_\hbar(F) + \zeta_\hbar \zeta_{g_2}^{\hbar}(F) \\
&= \zeta_{g_1 \hbar}(F) - \zeta_\hbar(F) + \zeta_{g_2 \hbar}(F) \ .
\end{aligned}
\tag{3.21}
$$

In the second last step we were able to apply the additivity of $\zeta_\hbar$ (3.16) since $\mathrm{supp}\,(\zeta_{g_j}^{\hbar}(F) - F) \subset \mathrm{supp}\,g_j$. Inserting $F = V(A)$ yields the claim in the proposition.

$\square$

We now insert (3.13) into (3.12) and find

$$
S\big(V(A^g) + \langle \Phi, \square_{A^g}(g\varphi) \rangle \big)\big|_{\phi=0} = S\big(V(A) + \langle \Phi, \square_A \varphi \rangle \big)\big|_{\phi=0}\, e^{i\mathfrak{G}(g,A)}.
\tag{3.22}
$$

Hence, using (3.7) and (3.2), we obtain the following action of $\mathcal{G}_o$ on the effective action:

$$
\Gamma(A^g, g\varphi) = \Gamma(A, \varphi) + \mathfrak{G}(g, A)\,.
\tag{3.23}
$$

Since $(A^{\hbar})^g = A^{g\hbar}$ and $\mathfrak{G}(g, A^{\hbar})$ does not depend on φ, we immediately see that $\mathfrak{G}$ satisfies the cocycle relation

$$
\mathfrak{G}(g\hbar, A) = \mathfrak{G}(g, A^{\hbar}) + \mathfrak{G}(\hbar, A)\ .
\tag{3.24}
$$

The Wess-Zumino consistency relation is just the infinitesimal version of this cocycle relation with

$$
G(X, A) = \frac{d}{d\lambda}\Big|_{\lambda=0} \mathfrak{G}(\exp(-\lambda X), A)\ .
\tag{3.25}
$$

This follows from the fact that $\mathcal{G}_o$ acts on Γ as an antirepresentation. To see it directly, we compute

$$
\begin{aligned}
\partial_X^A G(Y, A) - \partial_Y^A G(X, A) &= \frac{\partial^2}{\partial\lambda\partial\mu}\Big|_{\lambda=\mu=0} \Big(\mathfrak{G}\big(\exp(-\mu Y), A^{\exp(-\lambda X)}\big) - \mathfrak{G}\big(\exp(-\lambda X), A^{\exp(-\mu Y)}\big) \Big) \\
&= \frac{\partial^2}{\partial\lambda\partial\mu}\Big|_{\lambda=\mu=0} \Big(\mathfrak{G}\big(\exp(-\mu Y)\exp(-\lambda X), A\big) - \mathfrak{G}\big(\exp(-\lambda X)\exp(-\mu Y), A\big) \Big)\,,
\end{aligned}
\tag{3.26}
$$

where we used the cocycle condition (3.24) and the fact that the terms which depend only on one of the variables, λ or μ, do not contribute to the derivative. We use now the following consequences of the Baker-Campbell-Hausdorff formula:

$$
\exp(-\lambda X)\exp(-\mu Y) = g_+(\lambda, \mu)\exp(\tfrac{1}{2}\lambda\mu[X, Y])
\tag{3.27}
$$

and

$$
\exp(-\mu Y)\exp(-\lambda X) = g_-(\lambda, \mu)\exp(-\tfrac{1}{2}\lambda\mu[X, Y])\ ,
\tag{3.28}
$$

where g_+ and g_- coincide up to 2nd order. Inserting this into the previous formula and using again the cocycle condition, together with $\mathcal{G}(e, A) \equiv 0$, we find

$$
\begin{aligned}
\partial_X^A G(Y, A) - \partial_Y^A G(X, A) = \frac{\partial^2}{\partial \lambda \partial \mu}\Big|_{\lambda = \mu = 0} &\Big(\mathfrak{G}\big(g_-(\lambda, \mu), A^{\exp(-\frac{1}{2}\lambda\mu[X,Y])}\big) + \mathfrak{G}\big(\exp(-\tfrac{1}{2}\lambda\mu[X,Y]), A\big) \\
&- \mathfrak{G}\big(g_+(\lambda, \mu), A^{\exp(+\frac{1}{2}\lambda\mu[X,Y])}\big) - \mathfrak{G}\big(\exp(\tfrac{1}{2}\lambda\mu[X,Y]), A\big) \Big) \\
= &\ G([X, Y], A) ,
\end{aligned}
$$
(3.29)

since the terms involving g_+ and g_- cancel.

4. Consistency relation for the anomaly of the AMWI

We return to the more general framework introduced in Sect. 2. We aim at a derivation of a consistency relation for the anomaly map $X \mapsto \Delta X$ (2.25) of the AMWI (2.24) which holds for *general interactions*, by using only the AMWI. For this purpose we consider two paths g^λ and h^μ in $\mathcal{G}_c(M)$ with tangent vectors X and Y at 0, respectively. We compute

$$
\frac{\partial^2}{i \partial\lambda \partial\mu}\Big|_{\lambda = \mu = 0} \Big(S\big((g^\lambda h^\mu)_{L_q} F\big) - S\big((h^\mu g^\lambda)_{L_q} F\big) \Big) = S(F) \cdot_T \partial_{[X,Y]}(F + L_q) ,
$$
(4.1)

by using (2.22). According to the AMWI (2.24), it coincides for configurations ϕ with $\frac{\delta L}{\delta \phi}[\phi] = q$ with

$$
S(F) \cdot_T \Delta[X, Y](F) .
$$
(4.2)

Instead we can also use the AMWI after the first derivative and obtain on those configurations ϕ

$$
\begin{aligned}
\frac{\partial}{i\partial\lambda}\Big|_{\lambda=0} S\big((g^\lambda h^\mu)_{L_q} F\big) &= S(h^\mu_{L_q} F) \cdot_T \Delta X(h^\mu_{L_q} F) \\
&= \frac{d}{i dt}\Big|_{t=0} S\big(h^\mu_{L_q}(F + t(h^\mu_*)^{-1}\Delta X(h^\mu_L F))\big) ,
\end{aligned}
$$
(4.3)

where we used that ΔX is invariant under addition of an affine function of the field Φ. (This follows from the defining properties (P3) and (P5) of $\Delta X \in \mathrm{Lie}\,\mathscr{R}_c$, cf. Footnote 4.) Taking now the derivative with respect to μ and using again the AMWI we obtain

$$
\begin{aligned}
\frac{\partial^2}{i\partial\lambda\partial\mu}\Big|_{\lambda=\mu=0} S\big((g^\lambda h^\mu)_{L_q} F\big) &= \frac{d}{dt}\Big|_{t=0} S(F + t\Delta X(F)) \cdot_T \Big(\Delta Y(F + t\Delta X(F)) \\
&\qquad - t\partial_Y(\Delta X(F) + t\langle (\Delta X)'(F), \partial_Y(F + L)\rangle \Big) \\
&= S(F) \cdot_T \Big(i\Delta X(F) \cdot_T \Delta Y(F) + \langle \Delta Y'(F), \Delta X(F)\rangle \\
&\qquad - \partial_Y(\Delta X(F)) + \langle (\Delta X)'(F), \partial_Y(F + L)\rangle \Big)
\end{aligned}
$$
(4.4)

on the above mentioned configurations. We finally arrive at a consistency relation which does no longer depend on the source q and therefore holds for all configurations ϕ.

Theorem 4.1. *The anomaly Δ of the AMWI satisfies the consistency relation*

$$
\begin{aligned}
\Delta([X, Y])(F) &= \langle (\Delta Y)'(F), \Delta X(F)\rangle - \langle (\Delta X)'(F), \Delta Y(F)\rangle \\
&\quad + \partial_X(\Delta Y(F)) - \partial_Y(\Delta X(F)) \\
&\quad - \langle (\Delta Y)'(F), \partial_X(L + F)\rangle + \langle (\Delta X)'(F), \partial_Y(L + F)\rangle
\end{aligned}
$$
(4.5)

for $X, Y \in \mathrm{Lie}\,\mathcal{G}_c(M)$.

We call (4.5) the "extended Wess-Zumino consistency condition," because for *quadratic* functionals F, it reduces to the Wess-Zumino condition (3.10). Namely, for those functionals, $\Delta X(F)$ is a constant functional (see part *(iii)* of [8, Prop. 4.14]). Then $\partial_Y \Delta X(F)$ and[4] $\langle \Delta Y'(F), \Delta X(F) \rangle$ vanish for $X, Y \in \mathrm{Lie}\,\mathcal{G}_c(M)$, and the Wess-Zumino relation is obtained by using the identifications

$$G(X, A) = -\Delta X(V(A)) \tag{4.6}$$

and

$$\begin{aligned}
\partial_X^A G(Y, A) &\overset{(4.6)}{=} -\frac{d}{d\lambda}\Big|_{\lambda=0} \Delta Y\big(V(A^{g^\lambda})\big) \\
&\overset{(3.3)}{=} -\frac{d}{d\lambda}\Big|_{\lambda=0} \Delta Y\big(g_L^\lambda V(A)\big) \\
&= \langle \Delta Y'(V(A)), \partial_X\big(L + V(A)\big)\rangle\,,
\end{aligned} \tag{4.7}$$

where $g^\lambda \doteq \exp(-\lambda X)$ as in (3.8).

5. Consistency condition of the AMWI as a cocycle for Lie algebras

The consistency relation for Δ derived in the previous section actually shows that Δ is a Lie-algebraic cocycle. This can be most easily seen by starting from the UAMWI with its group theoretical cocycle ζ. The cocycle ζ intertwines two actions of $\mathcal{G}_c(M)$ on $\mathscr{F}_{\mathrm{loc}}(M)$, namely $(g, F) \mapsto g_L F$ and $(g, F) \mapsto g_L \zeta_g^{-1}(F)$. They induce representations R and P on the space of functions K on $\mathscr{F}_{\mathrm{loc}}(M)$ by

$$(R(g)K)(F) = K(g_L^{-1}F)\,, \quad (P(g)K)(F) = K(\zeta_g g_L^{-1}F) \tag{5.1}$$

The relation $R(g_1 g_2) = R(g_1)R(g_2)$ relies on $(g_1 g_2)_L = g_{1,L} g_{2,L}$; to obtain $P(g_1 g_2) = P(g_1)P(g_2)$ we additionally use our crucial input: the cocycle relation for ζ (2.15).

The corresponding representations r and p of $\mathrm{Lie}\,\mathcal{G}_c(M)$ act by derivations on smooth functions on $\mathscr{F}_{\mathrm{loc}}(M)$, *i.e.*

$$r(X)K(F) = \langle K'(F), -\partial_{XF} - \partial_X L \rangle\,, \quad p(X)K(F) = \langle K'(F), -\partial_X F - \partial_X L + \Delta X(F)\rangle, \tag{5.2}$$

where we used (2.25). Representations r and p differ by the linear map $X \mapsto q(X) = p(X) - r(X)$ with $q(X)K(F) = \langle K'(F), \Delta X(F)\rangle$. Since p and r are representations, q satisfies the relation

$$q([X, Y]) = [q(X), q(Y)] + [r(X), q(Y)] - [r(Y), q(X)] \tag{5.3}$$

It remains to compute the commutators of these derivations. We obtain

$$\begin{aligned}
(q(X)(q(Y)K))(F) &= \frac{d}{d\lambda}\Big|_{\lambda=0} (q(Y)K)(F + \lambda \Delta X(F)) \\
&= \frac{d}{d\lambda}\Big|_{\lambda=0} \langle K'(F + \lambda \Delta X(F)), \Delta Y(F + \lambda \Delta X(F))\rangle \\
&= \langle K''(F), \Delta X(F) \otimes \Delta Y(F)\rangle + \langle K'(F), \langle(\Delta Y)'(F), \Delta X(F)\rangle\rangle\,,
\end{aligned} \tag{5.4}$$

hence

$$\begin{aligned}
[q(X), q(Y)]K(F) &= \langle K'(F), \langle(\Delta Y)'(F), \Delta X(F)\rangle - \langle(\Delta X)'(F), \Delta Y(F)\rangle\rangle \\
&= \langle K'(F), [\Delta Y, \Delta X]_{\mathrm{Lie}\,\mathscr{R}_c}(F)\rangle\,,
\end{aligned} \tag{5.5}$$

where we use the explicit formula for the Lie bracket in $\mathrm{Lie}\,\mathscr{R}_c$ derived in [8, App. C]. Proceeding analogously to (5.4)-(5.5) we get

$$\begin{aligned}
[r(X), q(Y)]K(F) &= \langle K'(F), -\langle(\Delta Y)'(F), \partial_X F + \partial_X L\rangle + \partial_X(\Delta Y(F))\rangle \\
&= \langle K'(F), (\partial_X \Delta Y)(F)\rangle
\end{aligned} \tag{5.6}$$

---

[4]The defining property (P5) of $\Delta Y \in \mathrm{Lie}\,\mathscr{R}_c$ implies that $\Delta Y(F + c) = \Delta Y(F)$ for all $F \in \mathscr{F}_{\mathrm{loc}}(M)$, $c \in \mathbb{R}$.

with the representation $X \mapsto \partial_X$,

$$(\partial_X z)(F) \doteq \partial_X(z(F)) - \langle z'(F), \partial_X(F + L)\rangle \tag{5.7}$$

of $\mathrm{Lie}\,\mathcal{G}_c(M)$ by derivations on $\mathrm{Lie}\,\mathscr{R}_c$.[5] We also have the analogous relation with X and Y interchanged, so combining the two, we arrive at precisely the same consistency condition for the anomaly Δ of the AMWI as in Theorem 4.1 which now assumes the form:

Theorem 5.1. *The cocycle relation* (2.15) *for the anomaly* ζ *of the UMWI implies the following Lie-algebraic cocycle relation for the corresponding anomaly* Δ *of the MWI (i.e.,* Δ *is obtained from* ζ *by* (2.25)*):*

$$\Delta([X,Y])(F) = -[\Delta X, \Delta Y]_{\mathrm{Lie}\,\mathscr{R}_c}(F) + (\partial_X \Delta Y)(F) - (\partial_Y \Delta X)(F) \tag{5.8}$$

for $X, Y \in \mathrm{Lie}\,\mathcal{G}_c(M)$.

It is instructive to see how the seemingly different derivations in sections 4 and 5 lead to the same consistency relation for Δ. In the preceding section we solely used the definition of Δ in terms of the AMWI (2.24); here we solely used the expression of Δ in terms of ζ (2.25).

6. Infinitesimal cocycle condition from the nilpotency of the BV operator

In this section, we will derive the infinitesimal cocycle condition (5.8) within the BV formalism. The crucial insight is that the infinitesimal renormalisation group transformation ΔX, applied to a local functional F, can in fact be identified with the renormalized BV Laplacian $\triangle_F$ for the interaction F, applied to the vector field ∂_X,

$$\Delta X(F) = i \triangle_F \partial_X \ , \ \ X \in \mathrm{Lie}\,\mathcal{G}_c(M), \ F \in \mathscr{F}_{\mathrm{loc}}(M) \ , \tag{6.1}$$

where ∂_X is the vector field on $\mathscr{E}(M, \mathbb{R}^n)$ induced by X [18]. The operator $\triangle_F$ can be expressed by means of a generalization of the AMWI (see (6.15)), so the derivation of the anomaly consistency condition (4.5) given in the current section is essentially equivalent to the previous derivation in Sect. 4. Phrasing it in terms of the BV language, however, is important for showcasing the underlying algebraic structures naturally associated with the space of multivector fields and allows us to make connection with the literature, in particular [21,22]. Moreover, we see that the restriction to vector fields, which are polynomial of first order in the configuration ϕ, is crucial.

In the BV formalism one considers functions and vector fields on the configuration space as functions on the (-1)-shifted cotangent bundle $T^*[-1]\mathscr{E}(M, \mathbb{R}^n)$ over the configuration space. These functions form a graded commutative algebra $\mathrm{BV}(M)$, where the derivatives $\frac{\delta}{\delta\phi}$ are identified with the *antifields* $\Phi^{\ddagger}$,

$$\Phi_r^{\ddagger}(x)[dF[\phi]] = \frac{\delta F}{\delta\phi_r(x)}[\phi] \ , \ F \in \mathscr{F}(M) \ . \tag{6.2}$$

They are assumed to anticommute. The enlarged configuration space is $\tilde{\mathscr{E}}(M, \mathbb{R}^n) \doteq \mathscr{E}(M, \mathbb{R}^n) \oplus \mathscr{E}_{\mathrm{dens}}(M, \mathbb{R}^n)$, and the elements $\mathcal{F} \in \mathrm{BV}(M)$ are of the form

$$\mathcal{F} = \sum_{n,m} \langle f_{nm}, \Phi^{\otimes n} \otimes (\Phi^{\ddagger})^{\otimes m}\rangle \tag{6.3}$$

where the compactly supported distributions $f_{n,m}$ are symmetric in the first n and antisymmetric in the last m arguments. If $f_{nm} = 0$ for $m \neq 1$, the element $\mathcal{F}$ can be identified with a vector field on $\tilde{\mathscr{E}}(M, \mathbb{R}^n)$. The wave-front set conditions on f are the same as for the distributions characterizing elements of $\mathscr{F}(M)$. Analogously as for the functionals on the original configuration space, we introduce the spaces $\mathrm{BV}_{\mathrm{loc}}(M)$, $\mathrm{BV}_{n\,\mathrm{loc}}(M)$ and $\mathrm{BV}_{\bullet\,\mathrm{loc}}(M)$.

---

[5]To see that $X \to \partial_X$ is indeed a representation, note that it is the infinitesimal version of the representation D of $\mathcal{G}_c(M)$ on the space of maps $K : \mathscr{F}_{\mathrm{loc}}(M) \to \mathscr{F}_{\mathrm{loc}}(M)$ defined by $D(\mathscr{g})K(F) = \mathscr{g}_* K(\mathscr{g}_L^{-1} F)$.

The algebra $\mathrm{BV}_{\bullet\,\mathrm{loc}}(M)$ is equipped with a graded Poisson bracket, the *Schouten bracket*, also known as *antibracket*. For a functional $F \in \mathscr{F}_{\bullet\,\mathrm{loc}}(M)$ and a vector field $\mathcal{X} \in \mathrm{BV}_{\bullet\,\mathrm{loc}}(M)$ it is given by the action of the vector field on the functional as a derivation: $\{\mathcal{X}, F\} \doteq \mathcal{X}F$. For two vector fields we have $\{\mathcal{X}, \mathcal{Y}\} = [\mathcal{X}, \mathcal{Y}]$, *i.e.* the Lie bracket of vector fields, and for general elements $\mathcal{F}, \mathcal{G} \in \mathrm{BV}_{\bullet\,\mathrm{loc}}(M)$ we invoke the graded Leibniz rule.

In this notation, the antibracket takes the form:

$$\{\mathcal{F}, \mathcal{G}\} = \left\langle \frac{\delta^r \mathcal{F}}{\delta\phi}, \frac{\delta^l \mathcal{G}}{\delta\phi^{\ddagger}} \right\rangle - \left\langle \frac{\delta^r \mathcal{F}}{\delta\phi^{\ddagger}}, \frac{\delta^l \mathcal{G}}{\delta\phi} \right\rangle \tag{6.4}$$

where δ^r and δ^l signify right and left derivatives, respectively. We will use the convention that if no superscript is present, then the derivative is to be understood as the left derivative (see [18] for more detail).

The physical information about the equations of motion and symmetries of the classical theory with Lagrangian $L + F$ (L and F are both of vector field degree zero, we say that they don't depend on *antifields*), is encoded in the *classical BV operator*. For a compactly supported multivector field $\mathcal{X} \in \mathrm{BV}_{\bullet\,\mathrm{loc}}(M)$, we define

$$s_F \mathcal{X} \doteq \{\mathcal{X}, L(f) + F(f)\}, \tag{6.5}$$

where $f \equiv 1$ on $\mathrm{supp}\,\mathcal{X}$. To simplify the notation, we will often just write $\{\mathcal{X}, L + F\}$, unless we want to indicate a particular choice of the test function. As a consequence of the graded Jacobi identity for the Schouten bracket, s_F is a differential,

$$(s_F)^2(\mathcal{X}) = \{\{\mathcal{X}, L + F\}, L + F\} = \frac{1}{2}\{\mathcal{X}, \{L + F, L + F\}\} = 0, \tag{6.6}$$

since the Schouten bracket of two functionals, which do not depend on antifields, is zero. The space of on-shell functionals is formally encoded in the 0-th cohomology of the differential s_F, and the first cohomology gives the space of non-trivial (i.e. not vanishing on-shell) symmetries.

In quantum theory, the differential, in the absence of interaction *i.e.* $F = 0$, is deformed into

$$\hat{s}_0 \doteq T^{-1} \circ s_0 \circ T \tag{6.7}$$

with the time ordering operator T, extended to multilocal functionals of fields and antifields $\mathcal{F} \in \mathrm{BV}_{\bullet\,\mathrm{loc}}(M)$ such that the antifields are treated as classical sources, *i.e.* smooth densities. Since T is linear, also $\hat{s}_0$ is linear. Polynomials of linear local functionals have smooth functional derivatives. They form the subspace of regular functionals $\mathrm{BV}_{\mathrm{reg}}(M) \subset \mathrm{BV}_{\bullet\,\mathrm{loc}}(M)$. We compute

$$\hat{s}_0(\mathcal{F}) = (s_0 - i\Delta)(\mathcal{F}) \tag{6.8}$$

with the BV Laplacian

$$\triangle = \int_M \frac{\delta^2}{\delta\phi(x)\delta\phi^{\ddagger}(x)}. \tag{6.9}$$

Following [18], we define the renormalized BV Laplacian in the absence of interaction by

$$\triangle_0 \doteq i(\hat{s}_0 - s_0). \tag{6.10}$$

In the presence of an interaction $F \in \mathscr{F}_{\mathrm{loc}}(M)$ we define the *quantum BV operator* $\hat{s}_F$ by

$$\hat{s}_F(\mathcal{X}) \doteq e^{-iF}\hat{s}_0(e^{iF}\mathcal{X}), \ \mathcal{X} \in \mathrm{BV}_{\bullet\,\mathrm{loc}}(M) \tag{6.11}$$

and introduce the interaction-dependent BV Laplacian by

$$\triangle_F \doteq i(\hat{s}_F - s_F). \tag{6.12}$$

On regular functionals, $\triangle_F = \triangle_0 = \triangle$, but due to renormalization, the operators differ in general. Since $\hat{s}_0$ is linear, also $\hat{s}_F$ and $\triangle_F$ are linear. From (6.6) and (6.7) we immediately see that $(\hat{s}_0)^2 = 0$; hence, by (6.11), also $(\hat{s}_F)^2 = 0$.

The effects of renormalization can be understood by the AMWI which was extended to local multivector fields by Hollands and takes the form [22, Prop. 3]:

$$\hat{s}_F(e^{i\mathcal{X}}) = i\,e^{i\mathcal{X}}\left(\{\mathcal{X}, L+F\} + \frac{1}{2}\{\mathcal{X}, \mathcal{X}\} + A(F+\mathcal{X})\right),\tag{6.13}$$

where $\mathcal{X} = \sum \eta_i \mathcal{X}_i$, $F \in \mathscr{F}_{\mathrm{loc}}(M)$, $\mathcal{X}_i \in \mathrm{BV}_{\mathrm{loc}}(M)$ and η_i elements of a multiplier Grassmann algebra such that $\mathcal{X}$ is even[6]. A characterizes the anomalies. It is of the form

$$A(\mathcal{F}) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n(\mathcal{F}^n),\ \ \mathcal{F} \in \mathrm{BV}_{1\,\mathrm{loc}}(M)\ \text{even},\tag{6.14}$$

where $A_n : \mathrm{BV}_{n\,\mathrm{loc}}(M) \to \mathrm{BV}_{\mathrm{loc}}(M)$ are linear maps, which reduce the antifield number by 1, hence, $A(\mathcal{F})$ is odd. The relation between A and $\triangle_F$ is obtained from (6.12) and (6.13) by using that s_F is a derivation:

$$-i\triangle_F(e^{i\mathcal{X}}) = i\,e^{i\mathcal{X}}\left(\frac{1}{2}\{\mathcal{X}, \mathcal{X}\} + A(F+\mathcal{X})\right).\tag{6.15}$$

Taking into account that $\triangle_F$ is linear and $A(F) = 0$, this formula implies

$$\triangle_F(\mathcal{X}) = -i\frac{d}{d\lambda}\Big|_{\lambda=0}\triangle_F(e^{i\lambda\mathcal{X}}) = i\langle A'(F), \mathcal{X}\rangle.\tag{6.16}$$

Note that for $\mathcal{X} = \partial_X \eta$ the original AMWI (2.26) (or (2.24)) is obtained from the generalized AMWI (6.13) as the coefficient of η. Namely we get

$$T\big(e^{iF}\hat{s}_F(\partial_X)\big)[\phi] = s_0 \circ T\big(e^{iF}\partial_X\big)[\phi] = T\big(e^{iF}\partial_X\langle\Phi, q\rangle\big)[\phi]\tag{6.17}$$

with ϕ satisfying $q = \frac{\partial L}{\partial \phi}[\phi]$. This is similar to the result of [18], with the difference that here we introduced the external source q. In the last formula on the right-hand side, q can be pulled out from under the time-ordering operator and after one sets $q = \frac{\partial L}{\partial \phi}[\phi]$, one obtains the same relation between $\hat{s}_F$ and s_0 as in [18].

Now, applying AMWI (2.24) to the right-hand side, we obtain

$$T\big(e^{iF}\hat{s}_F(\partial_X)\big)[\phi] = T\Big(e^{iF}\big(\partial_X(L+F) - \Delta X(F)\big)\Big)[\phi].\tag{6.18}$$

Since $A(F) = 0$ this coincides with the corresponding term for the right hand side of (6.13), i.e., $\frac{d}{d\lambda}|_{\lambda=0}T\big(e^{i(F+\lambda\mathcal{X})}\big(\lambda\{\mathcal{X}, L+F\} + \frac{\lambda^2}{2}\{\mathcal{X}, \mathcal{X}\} + A(F+\lambda\mathcal{X})\big)\big)$, with the identification $\langle A'(F), \partial_X \eta\rangle = -\Delta X(F)\,\eta$, that is,

$$\Delta X(F) = -\frac{d^r}{d\eta}\Big|_{\eta=0} A(F + \partial_X \eta) \doteq -\langle A'(F), \partial_X\rangle,\tag{6.19}$$

hence by (6.16) we indeed obtain the announced relation (6.1) between ΔX and $\triangle_F$.

Hollands shows in [22, Prop.5] that the nilpotency of $\hat{s}_0$, i.e. $\hat{s}_0^2 = 0$, (or, equivalently, the nilpotency of $\hat{s}_F$, see (6.22) and (6.29) below) induces a consistency condition for the anomaly term in (6.13). We recall this result in Proposition 6.2 and provide an alternative (shorter) proof using a result of Fröb [21]: there is an L_∞-structure on $\mathrm{BV}_{1\,\mathrm{loc}}(M)$ underlying the AMWI (6.13). The brackets $[\bullet, \ldots, \bullet]_n^F : \mathrm{BV}_{1\,\mathrm{loc}}(M)^n \to \mathrm{BV}_{1\,\mathrm{loc}}(M)$ are linear and graded symmetric maps, given in terms of the generating function (for even $\mathcal{X}$) by

$$[e^{i\mathcal{X}}]^F \equiv \sum_{n=0}^{\infty} \frac{i^n}{n!}[\mathcal{X}, \ldots, \mathcal{X}]_n^F \doteq e^{-i\mathcal{X}}\hat{s}_F(e^{i\mathcal{X}}).\tag{6.20}$$

---

[6]For the use of external multipliers from Grassmann algebras (the η-trick [16]) see *e.g.* [7]. In particular note that, if η is odd, then $\eta\phi^\ddagger = -\phi^\ddagger\eta$.

Note that $[e^{i\mathcal{X}}]^F$ is odd. Obviously, with this definition, the AMWI (6.13) can be written as

$$[e^{i\mathcal{X}}]^F = i\big(\{\mathcal{X}, L+F\} + \tfrac{1}{2}\{\mathcal{X}, \mathcal{X}\} + A(F+\mathcal{X})\big) = i\big(\tfrac{1}{2}\{L+F+\mathcal{X}, L+F+\mathcal{X}\} + A(F+\mathcal{X})\big), \quad (6.21)$$

where $F \in \mathscr{F}_{\mathrm{loc}}(M)$. Crucially, we verify here (streamlining the argument of [21]) that the brackets defined by the formula (6.20) satisfy the generalized Jacobi identity, so we are indeed dealing with an L_∞ structure.

Proposition 6.1. *The brackets defined by* (6.20) *satisfy the generalized Jacobi identity:*

$$[e^{i\mathcal{X}}, [e^{i\mathcal{X}}]^F]^F = 0. \quad (6.22)$$

Proof. The result follows directly from the nilpotency of $\hat{s}_F$ and the fact that $\hat{s}_F(e^{i\mathcal{X}})$ is odd. To see this, first note that, for $\mathcal{G} \in \mathrm{BV}_{1\,\mathrm{loc}}(M)$ even, we obtain

$$[e^{i\mathcal{X}}, \mathcal{G}]^F = \frac{d}{i\,d\lambda}\Big|_{\lambda=0} [e^{i(\mathcal{X}+\lambda\mathcal{G})}]^F \overset{(6.20)}{=} -e^{-i\mathcal{X}}\mathcal{G}\,\hat{s}_F(e^{i\mathcal{X}}) + e^{-i\mathcal{X}}\hat{s}_F(e^{i\mathcal{X}}\mathcal{G}). \quad (6.23)$$

Inserting $\mathcal{G} = \eta\,[e^{i\mathcal{X}}]^F = \eta\,e^{-i\mathcal{X}}\hat{s}_F(e^{i\mathcal{X}})$ (with η an odd Grassmann variable) and omitting in the resulting formula the factor η, we get

$$[e^{i\mathcal{X}}, [e^{i\mathcal{X}}]^F]^F = -e^{-i2\mathcal{X}}\big(\hat{s}_F(e^{i\mathcal{X}})\big)^2 + e^{-i\mathcal{X}}\hat{s}_F^2(e^{i\mathcal{X}}) = 0. $$

$\square$

From (6.20) we see that the 0-bracket vanishes, $[-]_0^F = 0$, and that the 1-bracket is given by

$$[\mathcal{X}]_1^F = \hat{s}_F(\mathcal{X}) = s_F(\mathcal{X}) - i\triangle_F(\mathcal{X}) \quad (6.24)$$

(this formula is also immediately obtained from (6.21)), and from (6.21) we obtain for the 2-bracket

$$[\mathcal{X}, \mathcal{X}]_2^F = -i\big(\{\mathcal{X}, \mathcal{X}\} + \langle A''(F), \mathcal{X} \otimes \mathcal{X}\rangle\big) \quad (6.25)$$

and for the n-bracket (with $n > 2$)

$$[\mathcal{X}, \ldots, \mathcal{X}]_n^F = (-i)^{n-1}\langle A^{(n)}(F), \mathcal{X}^{\otimes n}\rangle. \quad (6.26)$$

With the L_∞ structure at hand, we come back to [22, Prop.5].

Proposition 6.2. *The anomaly map* $\mathcal{F} \mapsto A(\mathcal{F})$ *(where* $\mathcal{F} \in \mathrm{BV}_{1\,\mathrm{loc}}(M)$ *is even) defined by the generalized AMWI* (6.13) *satisfies the relation*

$$0 = \{L + \mathcal{F}, A(\mathcal{F})\} + \langle A'(\mathcal{F}), \big(\tfrac{1}{2}\{L + \mathcal{F}, L + \mathcal{F}\} + A(\mathcal{F})\big)\rangle. \quad (6.27)$$

Proof. We prove this proposition by verifying that the generalized Jacobi identity (6.22) is precisely the consistency condition (6.27) (which is not surprising since both rely on $\hat{s}_F^2 = 0$). To verify this, let $\mathcal{F} = F + \mathcal{X}$ with $\frac{\delta F}{\delta \phi^{\ddagger}} = 0$ and $\mathcal{X}$ even. Using that

$$[e^{i\mathcal{X}}, \mathcal{G}]^F = \frac{d^r}{i\,d\lambda}\Big|_{\lambda=0} [e^{i(\mathcal{X}+\mathcal{G}\lambda)}]^F \overset{(6.21)}{=} \{L + \mathcal{F}, \mathcal{G}\} + \langle A'(\mathcal{F}), \mathcal{G}\rangle \quad (6.28)$$

for $\mathcal{G} \in \mathrm{BV}_{1\,\mathrm{loc}}(M)$ odd and λ an odd Grassmann parameter, and applying again the AMWI (6.21), we obtain

$$0 = -i\,[e^{i\mathcal{X}}, [e^{i\mathcal{X}}]^F]^F \overset{(6.21)}{=} [e^{i\mathcal{X}}, \big(\tfrac{1}{2}\{L+\mathcal{F}, L+\mathcal{F}\} + A(\mathcal{F})\big)]^F$$

$$\overset{(6.28)}{=} \{L+\mathcal{F}, A(\mathcal{F})\} + \langle A'(\mathcal{F}), \big(\tfrac{1}{2}\{L+\mathcal{F}, L+\mathcal{F}\} + A(\mathcal{F})\big)\rangle,$$

where we also used the graded Jacobi identity for the antibracket; hence we arrive at (6.27). $\square$

Note that the vector fields ∂_X, $X \in \mathrm{Lie}\mathcal{G}_c(M)$ are of at most first order in ϕ. For these vector fields we have $\langle A''(F), \partial_X \otimes \partial_Y\rangle = 0$ (see [21] for a related result). To show this we start with the following lemma.

Lemma 6.3. *Let $\mathcal{G} \in T(\mathrm{BV}_{\bullet\,\mathrm{loc}})$ and let $\mathcal{Y} \in \mathrm{BV}_{1\,\mathrm{loc}}$ depend at most linearly on ϕ, then*

$$\left(\mathcal{G} \cdot \frac{\delta L}{\delta\phi(x)}\right) \cdot_T \mathcal{Y} = (\mathcal{G} \cdot_T \mathcal{Y}) \cdot \frac{\delta L}{\delta\phi(x)} + i\,\mathcal{G}\,\frac{\delta\mathcal{Y}}{\delta\phi(x)} \ , \tag{6.29}$$

Proof. Since $\mathcal{Y}$ is of first order in ϕ we have

$$\left(\mathcal{G} \cdot \frac{\delta L}{\delta\phi(x)}\right) \cdot_T \mathcal{Y} - \left(\mathcal{G} \cdot_T \mathcal{Y}\right)\frac{\delta L}{\delta\phi(x)} = \mathcal{G}\left\langle \frac{\delta}{\delta\phi}\frac{\delta L}{\delta\phi(x)}, E^{\mathrm{F}}\frac{\delta\mathcal{Y}}{\delta\phi} \right\rangle = i\,\mathcal{G}\,\frac{\delta\mathcal{Y}}{\delta\phi(x)} \tag{6.30}$$

where only the term with a single contraction with the Feynman propagator contributes. □

Next, we show that for elements of first order in ϕ, $\triangle_F$ acts as the unrenormalized BV Laplacian. In fact, this result holds not only for $F \in \mathscr{F}_{1\,\mathrm{loc}}$, but also for all even $\mathcal{F} \in \mathrm{BV}_{1\,\mathrm{loc}}$ satisfying the *quantum master equation* (QME), in the form proposed in [18].

Firstly, we define $\hat{s}_{\mathcal{F}}$ by means of (6.11), i.e. $\hat{s}_{\mathcal{F}}(\mathcal{X}) \doteq e^{-i\mathcal{F}}\hat{s}_0(e^{i\mathcal{F}}\mathcal{X})$, $\mathcal{X} \in \mathrm{BV}_{\bullet\,\mathrm{loc}}(M)$; and $\triangle_{\mathcal{F}}$ by means of (6.12), where $s_{\mathcal{F}}$ is defined analogously to (6.5). Note that also in this case $\hat{s}_{\mathcal{F}}^2 = 0$. However, if $\mathcal{F}$ contains antifields, it may happen that $s_{\mathcal{F}}$ (which is still a derivation) is no longer nilpotent, i.e. the *classical master equation*, $\{L + \mathcal{F}, L + \mathcal{F}\} = 0$, is violated. Instead, one only requires the QME, which is the condition that the formal S-matrix is invariant under s_0, i.e. [18]

$$s_0(Te^{i\mathcal{F}}) = 0\,. \tag{6.31}$$

Using an equivalent reformulation of the generalized AMWI (6.13) (i.e., the formulation given in [22, Prop. 3]), we can also rewrite the QME as:

$$s_0(Te^{i\mathcal{F}}) = iT\Big(\big(\tfrac{1}{2}\{L + \mathcal{F}, L + \mathcal{F}\} + A(\mathcal{F})\big)e^{i\mathcal{F}}\Big) = 0\,,$$

hence the QME (6.31) is equivalent to $\frac{1}{2}\{L+\mathcal{F}, L+\mathcal{F}\}+A(\mathcal{F}) = 0$. Note that (6.13) can equivalently be formulated in terms of an $\mathcal{F}$ containing antifields:

$$\hat{s}_{\mathcal{F}}(e^{i\mathcal{X}}) = i\,e^{i\mathcal{X}}\left(\frac{1}{2}\{\mathcal{X} + \mathcal{F} + L, \mathcal{X} + \mathcal{F} + L\} + A(\mathcal{F} + \mathcal{X})\right)\,, \tag{6.32}$$

and for $\mathcal{F}$ satisfying QME, this simplifies to

$$\hat{s}_{\mathcal{F}}(e^{i\mathcal{X}}) = i\,e^{i\mathcal{X}}\left(\{\mathcal{X}, L + \mathcal{F}\} + \frac{1}{2}\{\mathcal{X}, \mathcal{X}\} + A(\mathcal{F} + \mathcal{X}) - A(\mathcal{F})\right)\,, \tag{6.33}$$

Proposition 6.4. *Let $\mathcal{X}, \mathcal{Y} \in \mathrm{BV}_{1\,\mathrm{loc}}$ be of first order in ϕ and even (we multiply the usual vector fields with Grassman generators – the η-trick), and $\mathcal{F} \in \mathrm{BV}_{1\,\mathrm{loc}}$ even, such that (6.31) holds. Then*

$$\triangle_{\mathcal{F}}(\mathcal{X}\mathcal{Y}) = (\triangle_{\mathcal{F}}\mathcal{X})\mathcal{Y} + \mathcal{X}(\triangle_{\mathcal{F}}\mathcal{Y}) + \{\mathcal{X}, \mathcal{Y}\}\,. \tag{6.34}$$

Proof. Since $\triangle_{\mathcal{F}} = i(\hat{s}_{\mathcal{F}} - s_{\mathcal{F}})$ and $s_{\mathcal{F}}$ is a derivation, the statement is equivalent to

$$\hat{s}_{\mathcal{F}}(\mathcal{X}\mathcal{Y}) = \hat{s}_{\mathcal{F}}(\mathcal{X})\mathcal{Y} + \mathcal{X}\hat{s}_{\mathcal{F}}(\mathcal{Y}) - i\,\{\mathcal{X}, \mathcal{Y}\}\,. \tag{6.35}$$

Taking into account that $\hat{s}_{\mathcal{F}}(\mathcal{X}) = e^{-i\mathcal{F}}T^{-1}s_0(Te^{i\mathcal{F}}\mathcal{X})$ this implies

$$s_0(Te^{i\mathcal{F}}\mathcal{X}\mathcal{Y}) = s_0(Te^{i\mathcal{F}}\mathcal{X}) \cdot_T \mathcal{Y} + \mathcal{X} \cdot_T s_0(Te^{i\mathcal{F}}\mathcal{Y}) - i\,T(e^{i\mathcal{F}}\{\mathcal{X}, \mathcal{Y}\})\,, \tag{6.36}$$

by applying (2.4) and that $T^{-1}\mathcal{X} = \mathcal{X}$ (and similarly for $\mathcal{Y}$). Using the fact that $s_0 = -\langle \frac{\delta L}{\delta \phi}, \frac{\delta^r}{\delta \phi^{\ddagger}} \rangle$ (see (6.4)-(6.5)), we compute

$$
\begin{aligned}
s_0(Te^{i\mathcal{F}} & \mathcal{X}\mathcal{Y}) - s_0(Te^{i\mathcal{F}}\mathcal{X}) \cdot_T \mathcal{Y} - \mathcal{X} \cdot_T s_0(Te^{i\mathcal{F}}\mathcal{Y}) \\
&= -\Big\langle T\Big(e^{i\mathcal{F}}\big(\frac{\delta^r \mathcal{X}}{\delta \phi^{\ddagger}}\mathcal{Y} + \mathcal{X}\frac{\delta^r \mathcal{Y}}{\delta \phi^{\ddagger}} + i\frac{\delta^r \mathcal{F}}{\delta \phi^{\ddagger}}\mathcal{X}\mathcal{Y}\big)\Big), \frac{\delta L}{\delta \phi}\Big\rangle \\
&\quad + \Big\langle T\Big(e^{i\mathcal{F}}\frac{\delta^r \mathcal{X}}{\delta \phi^{\ddagger}}\Big), \frac{\delta L}{\delta \phi}\Big\rangle \cdot_T \mathcal{Y} + \mathcal{X} \cdot_T \Big\langle T\Big(e^{i\mathcal{F}}\frac{\delta^r \mathcal{Y}}{\delta \phi^{\ddagger}}\Big), \frac{\delta L}{\delta \phi}\Big\rangle \\
&\quad + \Big\langle T\Big(e^{i\mathcal{F}}i\frac{\delta^r \mathcal{F}}{\delta \phi^{\ddagger}}\mathcal{X}\Big), \frac{\delta L}{\delta \phi}\Big\rangle \cdot_T \mathcal{Y} + \mathcal{X} \cdot_T \Big\langle T\Big(e^{i\mathcal{F}}i\frac{\delta^r \mathcal{F}}{\delta \phi^{\ddagger}}\mathcal{Y}\Big), \frac{\delta L}{\delta \phi}\Big\rangle. \quad (6.37)
\end{aligned}
$$

Next we show that the terms containing $\frac{\delta^r \mathcal{F}}{\delta \phi^{\ddagger}}$ cancel out since $\mathcal{F}$ satisfies the QME (6.31). To do this, let $\mathcal{G}_x \doteq i\,T\big(e^{i\mathcal{F}}\frac{\delta^r \mathcal{F}}{\delta \phi^{\ddagger}(x)}\big)$. With that, the considered terms can be written as

$$
-\Big\langle \mathcal{G}_\bullet \cdot_T \mathcal{X} \cdot_T \mathcal{Y}, \frac{\delta L}{\delta \phi}\Big\rangle + \Big\langle (\mathcal{G}_\bullet \cdot_T \mathcal{X}), \frac{\delta L}{\delta \phi}\Big\rangle \cdot_T \mathcal{Y} + \mathcal{X} \cdot_T \Big\langle (\mathcal{G}_\bullet \cdot_T \mathcal{Y}), \frac{\delta L}{\delta \phi}\Big\rangle. \quad (6.38)
$$

By iterated use of (6.29) (Lemma 6.3) we obtain for the first term

$$
\begin{aligned}
\Big\langle (\mathcal{G}_\bullet \cdot_T \mathcal{X}) \cdot_T \mathcal{Y}, \frac{\delta L}{\delta \phi}\Big\rangle &= \Big\langle \mathcal{G}_\bullet, \frac{\delta L}{\delta \phi}\Big\rangle \cdot_T \mathcal{X} \cdot_T \mathcal{Y} \\
&\quad - i\Big\langle \mathcal{G}_\bullet, \frac{\delta \mathcal{X}}{\delta \phi}\Big\rangle \cdot_T \mathcal{Y} - i\mathcal{X} \cdot_T \Big\langle \mathcal{G}_\bullet, \frac{\delta \mathcal{Y}}{\delta \phi}\Big\rangle, \quad (6.39)
\end{aligned}
$$

where we also take into account that $\frac{\delta \mathcal{Y}}{\delta \phi}$ is of zeroth order in ϕ. Using again (6.29) we also obtain

$$
\Big\langle \mathcal{G}_\bullet, \frac{\delta L}{\delta \phi}\Big\rangle \cdot_T \mathcal{X} \cdot_T \mathcal{Y} = \Big\langle (\mathcal{G}_\bullet \cdot_T \mathcal{X}), \frac{\delta L}{\delta \phi}\Big\rangle \cdot_T \mathcal{Y} + i\Big\langle \mathcal{G}_\bullet, \frac{\delta \mathcal{X}}{\delta \phi}\Big\rangle \cdot_T \mathcal{Y}. \quad (6.40)
$$

Note that on the r.h.s. we may exchange $\mathcal{X}$ and $\mathcal{Y}$. Inserting these results into (6.38) and with $\langle \mathcal{G}_\bullet, \frac{\delta L}{\delta \phi}\rangle = -s_0(Te^{i\mathcal{F}})$ we get

$$
(6.38) = -s_0(Te^{i\mathcal{F}}) \cdot_T \mathcal{X} \cdot_T \mathcal{Y} = 0 \quad (6.41)
$$

by the QME.

To further transform the remaining terms in (6.37), we apply again Lemma 6.3

$$
\begin{aligned}
(6.37) &= i\Big\langle T\Big(e^{i\mathcal{F}}\frac{\delta^r \mathcal{X}}{\delta \phi^{\ddagger}}\Big), \frac{\delta \mathcal{Y}}{\delta \phi}\Big\rangle + i\Big\langle \frac{\delta \mathcal{X}}{\delta \phi}, T\Big(e^{i\mathcal{F}}\frac{\delta^r \mathcal{Y}}{\delta \phi^{\ddagger}}\Big)\Big\rangle \\
&= i\,Te^{i\mathcal{F}}\Big(\Big\langle \frac{\delta \mathcal{X}}{\delta \phi}, \frac{\delta^r \mathcal{Y}}{\delta \phi^{\ddagger}}\Big\rangle + \Big\langle \frac{\delta^r \mathcal{X}}{\delta \phi^{\ddagger}}, \frac{\delta \mathcal{Y}}{\delta \phi}\Big\rangle\Big) \\
&\equiv -i\,T(e^{i\mathcal{F}}\{\mathcal{X}, \mathcal{Y}\}), \quad (6.42)
\end{aligned}
$$

where in the 2nd equation we used again that $\frac{\delta \mathcal{X}}{\delta \phi}$ and $\frac{\delta \mathcal{Y}}{\delta \phi}$ are of zeroth order in ϕ and, hence, can be shifted into the argument of T. $\qquad\square$

The result on A'' follows now directly from Proposition 6.4.

Proposition 6.5. *Let A be the anomaly appearing in the AMWI. Let $\mathcal{F} \in \mathrm{BV}_{1\,\mathrm{loc}}(M)$ even, satisfying QME (6.31) and let $\mathcal{X}, \mathcal{Y} \in \mathrm{BV}_{1\,\mathrm{loc}}(M)$ be at most linear in ϕ. Then*

$$
\langle A''(\mathcal{F}), \mathcal{X} \otimes \mathcal{Y}\rangle = 0. \quad (6.43)
$$

Proof. Without restriction of generality we may assume that both $\mathcal{X}$ and $\mathcal{Y}$ are even (by using the η-trick). Let $\lambda, \mu \in \mathbb{R}$. From the generalized AMWI (6.13) we have that:

$$\hat{s}_{\mathcal{F}}(e^{i(\lambda\mathcal{X}+\mu\mathcal{Y})}) \equiv (s_{\mathcal{F}} - i\triangle_{\mathcal{F}})(e^{i(\lambda\mathcal{X}+\mu\mathcal{Y})}) \tag{6.44}$$

$$\stackrel{(6.33)}{=} i\,e^{i(\lambda\mathcal{X}+\mu\mathcal{Y})}\left(\{\lambda\mathcal{X} + \mu\mathcal{Y}, L + \mathcal{F}\} + \frac{1}{2}\{\lambda\mathcal{X} + \mu\mathcal{Y}, \lambda\mathcal{X} + \mu\mathcal{Y}\} + A(\mathcal{F} + \lambda\mathcal{X} + \mu\mathcal{Y}) - A(\mathcal{F})\right)$$

Selecting the terms proportional to $\lambda\mu$ we obtain

$$(s_{\mathcal{F}} - i\triangle_{\mathcal{F}})(\mathcal{X}\mathcal{Y}) \tag{6.45}$$
$$= \mathcal{X}\big(\{\mathcal{Y}, L + \mathcal{F}\} - i\triangle_{\mathcal{F}}(\mathcal{Y})\big) + \mathcal{Y}\big(\{\mathcal{X}, L + \mathcal{F}\} - i\triangle_{\mathcal{F}}(\mathcal{X})\big) - i\{\mathcal{X}, \mathcal{Y}\} - i\langle A''(\mathcal{F}), \mathcal{X} \otimes \mathcal{Y}\rangle,$$

by using the analog of (6.16) for $\mathcal{F} \in \mathrm{BV}_{1\,\mathrm{loc}}(M)$. The statement follows from the derivation property of $s_{\mathcal{F}}$ and the preceding proposition. $\qquad\square$

For $F \in \mathscr{F}_{1\,\mathrm{loc}}(M)$, one obtains a map

$$\mathrm{Lie}\,\mathcal{G}_c(M) \times \mathscr{F}_{\mathrm{loc}}(M) \ni (X, F) \mapsto s_F(\partial_X) - i\triangle_F(\partial_X) = \partial_X(F + L) - \Delta X(F), \tag{6.46}$$

which coincides with the action previously constructed in (5.2). The fact that it is an action was derived from the cocycle relation (5.8) for $X \mapsto \Delta X$ as a consequence of the cocycle relation for the anomaly map ζ in the UAMWI.

Actually, the cocycle relation for Δ in the form of the equivalent consistency relation (4.5) derives directly from the BV-consistency condition (6.27).

Proposition 6.6. *The BV consistency relation implies the extended Wess-Zumino consistency relation* (4.5).

Proof. We insert $\mathcal{F} = F + \partial_{X_1}\eta_1 + \partial_{X_2}\eta_2$ into the BV consistency relation (6.27), where $F \in \mathscr{F}_{\mathrm{loc}}(M)$ and η_1, η_2 are Grassmann generators. As before, we use the fact that $\langle A''(F), \partial_{X_1} \otimes \partial_{X_2}\rangle = 0$. Since $A(F) = 0$ we obtain the following finite Taylor expansion in η_1, η_2:

$$A(\mathcal{F}) = -\Delta X_1(F)\,\eta_1 - \Delta X_2(F)\,\eta_2, \tag{6.47}$$

where we also used (6.19). In particular note that $\frac{\delta A(\mathcal{F})}{\delta\phi^{\ddagger}} = 0$. With that we obtain

$$\{L + \mathcal{F}, A(\mathcal{F})\} = -\{(\partial_{X_1}\eta_1 + \partial_{X_2}\eta_2), \big(\Delta X_1(F)\,\eta_1 + \Delta X_2(F)\,\eta_2\big)\}$$
$$= \big(-\partial_{X_1}\Delta X_2(F) + \partial_{X_2}\Delta X_1(F)\big)\,\eta_1\eta_2. \tag{6.48}$$

Note that

$$\langle A'(\mathcal{F}), (G + \partial_Z\eta)\rangle = \frac{d}{d\tau}\Big|_{\tau=0} A\big(\mathcal{F} + \tau(G + \partial_Z\eta)\big) \tag{6.49}$$
$$= -\frac{d}{d\tau}\Big|_{\tau=0}\Big(\Delta X_1(F + \tau G)\,\eta_1 + \Delta X_2(F + \tau G)\,\eta_2 + \tau\Delta Z(F + \tau G)\,\eta\Big)$$
$$= -\langle(\Delta X_1)'(F), G\rangle\eta_1 - \langle(\Delta X_2)'(F), G\rangle\eta_2 - \Delta Z(F)\,\eta, \tag{6.50}$$

where $G \in \mathscr{F}_{\mathrm{loc}}(M)$, $Z \in \mathrm{Lie}\,\mathcal{G}_c(M)$ and η is another Grassmann generator. Hence, using (2.22), we obtain

$$\langle A'(\mathcal{F}), \big(\tfrac{1}{2}\{L + \mathcal{F}, L + \mathcal{F}\}\big)\rangle = \langle A'(\mathcal{F}), \big((\partial_{X_1}\eta_1 + \partial_{X_2}\eta_2)(L + F) - \partial_{[X_1, X_2]}\,\eta_1\eta_2\big)\rangle$$
$$= \Big(-\langle(\Delta X_1)'(F), \partial_{X_2}(L + F)\rangle + \langle(\Delta X_2)'(F), \partial_{X_1}(L + F)\rangle + \Delta[X_1, X_2](F)\Big)\eta_1\eta_2 \tag{6.51}$$

and

$$\langle A'(\mathcal{F}),\, A(\mathcal{F})\rangle = -\langle A'(\mathcal{F}),\, (\Delta X_1(F)\,\eta_1 + \Delta X_2(F)\,\eta_2)\rangle$$

$$= \Big(\langle(\Delta X_1)'(F), \Delta X_2(F)\rangle - \langle(\Delta X_2)'(F), \Delta X_1(F)\rangle\Big)\eta_1\eta_2\,. \qquad (6.52)$$

Composing (6.48), (6.51) and (6.52), we obtain the consistency equation (4.5). $\qquad\square$

Remark 6.7. Note that by tracing back the arguments given in this section, we can see that Proposition 6.6 essentially states that the generalized Wess-Zumino consistency condition is the consequence of $\hat{s}_F^2 = 0$. We can compare this with a simple fact that the nilpotency of the non-renormalized BV Lapalacian $\triangle$ (see (6.9)) implies an analogous statement for vector fields. Without the loss of generality, we assume $\mathcal{X}$ and $\mathcal{Y}$ to be even (we multiply the usual vector fields with Grassman parameters) $\mathcal{X}, \mathcal{Y} \in \mathrm{BV}_{\mathrm{reg}}(M)$ (regular multivector fields):

$$0 = \triangle^2(\mathcal{X}\mathcal{Y}) = \triangle\big((\triangle\mathcal{X})\mathcal{Y} + \mathcal{X}(\triangle\mathcal{Y}) + \{\mathcal{X}, \mathcal{Y}\}\big) = \partial_{\mathcal{X}}(\triangle\mathcal{Y}) + \partial_{\mathcal{Y}}(\triangle\mathcal{X}) + \triangle(\{\mathcal{X}, \mathcal{Y}\})$$

by using that Δ satisfies a relation analogous to (6.34), where $\partial_{\mathcal{X}}$, $\partial_{\mathcal{Y}}$ denotes the natural action of vector fields on functionals as derivations. $\qquad\maltese$

7. Summary and Outlook

For the symmetries $g \in \mathcal{G}_c(M)$ considered in this paper, we have proven that the anomaly map Δ in AMWI satisfies a consistency condition stated in Thm. 4.1, which we named *extended Wess-Zumino condition*, as it can be understood as an extension to non-quadratic interactions of the well-known Wess-Zumino consistency condition (3.10). Our proof uses only the AMWI. In contrast to an analogous procedure in [22], our extended Wess-Zumino consistency condition and its derivation do not need the antifield formalism.

We then showed that our extended Wess-Zumino consistency condition can be deduced from the cocycle relation (2.15) for the anomaly map ζ occuring in the UAMWI and, hence, describes a Lie algebraic cocycle of the Lie algebra of $\mathcal{G}_c(M)$ with values in the Lie algebra of the Stückelberg-Petermann renormalization group (Thm. 5.1). Conversely, in the framework of perturbation theory, starting with the AMWI one can derive the UAMWI with an anomaly map ζ fulfilling the cocycle relation (2.15), see Thm. A.1.

We also investigated the connection to the BV formalism (as previously studied in [18, 21, 22]) and the underlying algebraic structures. In particular we verified that the extended Wess-Zumino consistency condition (4.5) can be obtained from the nilpotency of the BV operator $\hat{s}_F$, by restricting to symmetries $g \in \mathcal{G}_c(M)$ (Prop. 6.6). Our proof starts with the consistency condition (6.27) (Prop. 6.2) proven by Hollands [22], which can be understood as the the generalized Jacobi identity for the underlying L_∞-algebras (see the proof of Prop. 6.2) and relies on $\hat{s}_F^2 = 0$.

It is an interesting open problem to find the grouplike structure associated to the L_∞-structure for more general symmetries and to understand its relation to renormalization. This will be addressed in our future work.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Appendix A. Off shell UAMWI

In [8, Thm. 10.3(i)] we showed that in formal perturbation theory the UAMWI holds on shell. In this appendix we prove by a slightly improved argument that also the off-shell version of the UAMWI 2.16 holds. Here "off-shell" means that ϕ can be arbitrary, but we also introduce external sources q and at the end we set $q = \frac{\delta L}{\delta \phi}[\phi]$, so *a priori* our expressions are functions of two variables,

ϕ and q and the UAMWI holds on a subspace where the condition $q = \frac{\delta L}{\delta \phi}[\phi]$ is satisfied. Crucially, on that subspace the left-hand side of UAMWI proven below does not depend on q.

Theorem A.1. *In formal perturbation theory, the AMWI implies the off-shell unitary AMWI*

$$S \circ \zeta_g(F)[\phi] = S \circ g_{L_q}(F)[\phi] \ , \quad for \ \ \phi \ solving \ q = \frac{\delta L}{\delta \phi}[\phi] \ , \quad for \ all \ F \in \mathscr{F}_{\mathrm{loc}}(M) \ , g \in \mathcal{G}_c(M) \ ,$$
(A.1)

with a cocycle ζ taking values in $\mathscr{R}_c$ and with $\operatorname{supp}\zeta_g \subset \operatorname{supp}g$.

Proof. Since the elements of $\mathcal{G}_c(M)$ have compact support and depend smoothly on x, $\mathcal{G}_c(M)$ must be connected. Therefore, given any $g \in \mathcal{G}_c(M)$, there exists a smooth curve $\lambda \mapsto g^\lambda \in \mathcal{G}_c(M)$ with $g^0 = e$ and $g^1 = g$. Let $X^\lambda \in \operatorname{Lie}\mathcal{G}_c(M)$ be defined by $\frac{d}{d\lambda}g^\lambda = X^\lambda g^\lambda$.

In the next step, want to find a smooth curve $\lambda \mapsto \zeta_{g^\lambda}^{-1} \in \mathscr{R}_c$ with $\zeta_e^{-1} = \mathrm{id}$ and

$$\frac{d}{d\lambda}S\big(g_{L_q}^\lambda \zeta_{g^\lambda}^{-1}(F)\big)[\phi] = 0 \text{ with } q = \frac{\delta L}{\delta \phi}[\phi] \ .$$
(A.2)

Note that inserting $\lambda = 0$ and $\lambda = 1$ into $S \circ g_{L_q}^\lambda \circ \zeta_{g^\lambda}^{-1}$, we obtain the unitary AMWI (2.16).

To search for the desired curve, we will first derive a differential equation that it has to solve. Note that for $G \in \mathscr{F}_{\mathrm{loc}}(M)$, we have

$$\frac{d}{d\lambda}g_{L_q}^\lambda G = \partial_{X^\lambda}g_{L_q}^\lambda G + \partial_{X^\lambda}(L_q) \ ,$$
(A.3)

so by using this result, we perform the differentiation in (A.2) and obtain the condition

$$S\big(g_{L_q}^\lambda \zeta_{g^\lambda}^{-1}(F)\big) \cdot_T \left(\partial_{X^\lambda}g_{L_q}^\lambda \zeta_{g^\lambda}^{-1}(F) + \partial_{X^\lambda}(L_q) + g_*^\lambda \frac{d}{d\lambda}\zeta_{g^\lambda}^{-1}(F)\right)[\phi] = 0 \ \text{ for } q = \frac{\delta L}{\delta \phi}[\phi] \ .$$
(A.4)

We insert the anomalous MWI (2.24) and find

$$S\big(g_{L_q}^\lambda \zeta_{g^\lambda}^{-1}(F)\big) \cdot_T \left(\Delta X^\lambda\big(g_{L_q}^\lambda \zeta_{g^\lambda}^{-1}(F)\big) + g_*^\lambda \frac{d}{d\lambda}\zeta_{g^\lambda}^{-1}(F)\right)[\phi] = 0 \text{ for } q = \frac{\delta L}{\delta \phi}[\phi] \ .$$
(A.5)

On the other hand, $\Delta X^\lambda \circ g_{L_q}^\lambda = \Delta X^\lambda \circ g_L^\lambda$, since $\delta_{g^\lambda}\langle \Phi, q\rangle$ is at most of first order in Φ and due to the defining property (*iii*) of $\operatorname{Lie}\mathscr{R}_c$. We thus get the desired family $\lambda \mapsto \zeta_{g^\lambda}^{-1}$ as the unique solution of the differential equation

$$\frac{d}{d\lambda}\zeta_{g^\lambda}^{-1} = -(g_*^\lambda)^{-1}\Delta X^\lambda g_L^\lambda \zeta_{g^\lambda}^{-1} \ ,$$
(A.6)

with the initial condition $\zeta_{g^0}^{-1} = \mathrm{id}$. As explained for the case $q = 0$,

$$(g_*^\lambda)^{-1}\Delta X^\lambda g_L^\lambda \in \operatorname{Lie}\mathscr{R}_c$$
(A.7)

holds, hence $\zeta_{g^\lambda}^{-1} \in \mathscr{R}_c$ follows, so in particular $\zeta_g \in \mathscr{R}_c$. Since the differential equation (A.6) determining ζ does not contain q, we explicitly see that ζ can be chosen such that it does not depend on q either. Hence, the proof of $\operatorname{supp}\zeta_g \subset \operatorname{supp}g$ can be adopted from the case $q = 0$ as it stands.

It remains to show that ζ satisfies the cocycle identity. Applying three times the UAMWI (2.16) we obtain

$$\begin{aligned}
S \circ \zeta_{gh}(F)[\phi] &= S \circ (gh)_{L_q}(F)[\phi] = S \circ g_{L_q} \circ h_{L_q}(F)[\phi] \\
&= S \circ \zeta_g \circ h_{L_q}(F)[\phi] = S \circ h_{L_q} \circ (h_{L_q}^{-1}\zeta_g h_{L_q})(F)[\phi] \\
&= S \circ \zeta_h \circ (h_{L_q}^{-1}\zeta_g h_{L_q})(F)[\phi]
\end{aligned}$$
(A.8)

for ϕ solving $q = \frac{\delta L}{\delta \phi}[\phi]$. Using again that $\delta_\hbar \langle \Phi, q \rangle$ is at most of first order in Φ, we conclude that

$$\zeta_g \hbar_{L_q}(F) = \zeta_g \big(\hbar_L F - \delta_\hbar \langle \Phi, q \rangle \big) = \zeta_g \hbar_L(F) - \delta_\hbar \langle \Phi, q \rangle, \tag{A.9}$$

since $\zeta_g \in \mathscr{R}_c$.[7] With that we obtain

$$\hbar_{L_q}^{-1} \zeta_g \hbar_{L_q} = \hbar_L^{-1} \zeta_g \hbar_L \equiv \zeta_g^\hbar, \tag{A.10}$$

hence the cocycle relation

$$S \circ \zeta_{g\hbar}(F) = S \circ \zeta_\hbar \circ (\zeta_g)^\hbar(F) \tag{A.11}$$

holds for all field configurations ϕ. Since the *off-shell* S-matrix is injective, we obtain the cocycle relation for ζ. $\qquad\qquad\square$

References

[1] L. Alvarez-Gaumé "An Introduction to Anomalies," in *Fundamental Problems of Gauge Field Theory*, Proceedings of the 6Th Course of The International School Of Mathematical Physics, Erice, Italy, July 1-14, 1985, Editors: G. Velo, A.S. Wightman, NATO Sci.Ser.B **141** (1986) pp.93-206.

[2] W. A. Bardeen and B. Zumino, "Consistent and covariant anomalies in gauge and gravitational theories," Nucl. Phys. **B244** (1984) 421-453.

[3] R. A. Bertlmann, *Anomalies in Quantum Field Theory*, Oxford University Press Inc., New York, 1996.

[4] F. Brennecke and M. Dütsch, "Removal of violations of the Master Ward Identity in perturbative QFT," Rev. Math. Phys. **20** (2008) 119.

[5] F. Brennecke and M. Dütsch, "The quantum action principle in the framework of causal perturbation theory," in *Quantum Field Theory: Competitive Models*, B. Fauser, J. Tolksdorf and E. Zeidler, eds., Birkhäuser, Basel, 2009; pp. 177–196.

[6] R. Brunetti, M. Dütsch and K. Fredenhagen, "Perturbative algebraic quantum field theory and the renormalization groups," Adv. Theor. Math. Phys. **13** (2009) 1541-1599.

[7] R. Brunetti, M. Dütsch, K. Fredenhagen and K. Rejzner, "C^*-algebraic approach to interacting quantum field theory: Inclusion of Fermi fields," [arXiv:2103.05740 [math-ph]], Lett. Math. Phys. **112**, 101 (2022), https://doi.org/10.1007/s11005-022-01590-7 (2022).

[8] R. Brunetti, M. Dütsch, K. Fredenhagen and K. Rejzner "The unitary Master Ward Identity: time slice axiom, Noether's theorem and anomalies," Ann. Henri Poincaré (2022), https://doi.org/10.1007/s00023-022-01218-5.

[9] R. Brunetti and K. Fredenhagen, "Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds," Commun. Math. Phys. **208** (2000) 623-661.

[10] R. Brunetti, K. Fredenhagen and R. Verch, "The Generally covariant locality principle: A New paradigm for local quantum field theory," Commun. Math. Phys. **237** (2003) 31-68.

[11] D. Buchholz and K. Fredenhagen, "A C*-algebraic approach to interacting quantum field theories," Commun. Math. Phys. **377** (2020), 947–969.

[12] M. Dütsch and F.-M. Boas, "The Master Ward Identity," Rev. Math. Phys. **14** (2002), 977–1049.

[13] M. Dütsch and K. Fredenhagen, "Algebraic Quantum Field Theory, Perturbation Theory and the Loop Expansion," Commun. Math. Phys. **219** (2001), 5–30.

[14] M. Dütsch and K. Fredenhagen, "The Master Ward Identity and generalized Schwinger–Dyson equation in classical field theory," Commun. Math. Phys. **243** (2003), 275–314.

[15] M. Dütsch and K. Fredenhagen, "Causal perturbation theory in terms of retarded products, and a proof of the Action Ward Identity," Rev. Math. Phys. **16** (2004), 1291–1348.

[16] M. Dütsch, *From Classical Field Theory to Perturbative Quantum Field Theory*, Progress in Mathematical Physics **74**, Birkhäuser, Cham, 2019.

[17] H. Epstein and V. Glaser, "The Role of locality in perturbation theory," Annales Poincare Phys. Theor. A **19** (1973) 211-295.

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[7]The defining property of $Z \in \mathscr{R}_c$ corresponding to the renormalization condition "off shell field equation" for the timeordered product (2.7) reads $Z(F + \langle \Phi, q \rangle) = Z(F) + \langle \Phi, q \rangle$, see e.g. [15] and cf. the defining property (P3) of $\text{Lie}\,\mathscr{R}_c$. In addition, $Z(F + c) = Z(F) + c$ for $c \in \mathbb{R}$ follows from the compactness of supp Z, see [8, Def. 4.2] and cf. the defining property (P5) of $\text{Lie}\,\mathscr{R}_c$.

[18] K. Fredenhagen and K. Rejzner, "Batalin–Vilkoviski formalism in perturbative algebraic quantum field theory," Commun. Math. Phys. **317** (2013), 697–725.

[19] K. Fredenhagen and K. Rejzner, "Perturbative Construction of Models of Algebraic Quantum Field Theory," in *Advances in Algebraic Quantum Field Theory*, R. Brunetti, C. Dappiaggi, K. Fredenhagen and J. Yngvason eds., Mathematical Physics Studies, Springer 2015; pp. 31–74.

[20] K. Fredenhagen and K. Rejzner, "Quantum field theory on curved spacetimes: Axiomatic framework and examples," J. Math. Phys. **57** (2016) no.3, 031101.

[21] M. Fröb, "Anomalies in Time-Ordered Products and Applications to the BV–BRST Formulation of Quantum Gauge Theories," Commun. Math. Phys. **372** (2019), 281–341.

[22] S. Hollands, "Renormalized Quantum Yang-Mills Fields in Curved Spacetime," Rev. Math. Phys. **20** (2008), 1033–1172.

[23] S. Hollands and R.M. Wald, "Local Wick polynomials and time ordered products of quantum fields in curved space-time," Commun. Math. Phys. **223** (2001) 289-326.

[24] S. Hollands and R.M. Wald, "Existence of local covariant time-ordered products of quantum fields in curved spacetime," Commun. Math. Phys. **231** (2002), 309–345.

[25] G. Popineau and R. Stora, "A pedagogical remark on the main theorem of perturbative renormalization theory," Nucl. Phys. B **912** (2016), 70–78, preprint: LAPP–TH, Lyon (1982).

[26] K. Rejzner, "Remarks on Local Symmetry Invariance in Perturbative Algebraic Quantum Field Theory," Annales Henri Poincaré **16** (2014), 205–238.

[27] A. Schenkel and J. Zahn, "Global Anomalies on Lorentzian Space-Times," Ann. H. Poincaré, **18** (2017), 2693–2714.

[28] R. Stora, "Algebres différentielles en théorie des champs," Ann. l'institut Fourier, **37(4)** (1987), 235–245.

[29] J. Wess and B. Zumino, "Consequences Of Anomalous Ward Identities," Phys. Lett. B37 (1971), 95.

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