Operator-algebraic renormalization and wavelets

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(Dated: February 5, 2020)

We report on a rigorous operator-algebraic renormalization group scheme and construct the continuum free field as the scaling limit of Hamiltonian lattice systems using wavelet theory. A renormalization group step is determined by the scaling equation identifying lattice observables with the continuum field smeared by compactly supported wavelets. Causality follows from Lieb-Robinson bounds for harmonic lattice systems. The scheme is related with the multi-scale entanglement renormalization ansatz and augments the semi-continuum limit of quantum systems.

arXiv:2002.01442v1 [math-ph] 4 Feb 2020

Lattice regularization is a standard procedure to define continuum quantum field theories. While the classical works of Glimm-Jaffe and others [1] have rigorously constructed interacting models, the lattice and continuum theories are related rather indirectly in terms of correlation functions. A recent attempt in conformal field theory (CFT) inspired by the block-spin transformation resulted in a discontinuous action of symmetries, even the translations [2–5]. Here, we utilize an operator-algebraic approach to the renormalization group for lattice field theories clarifying the roles of scaling maps and states ([6] for details and proofs). We take scaling maps based on compactly supported Daubechies wavelets [7]. Renormalizing the ground states of lattice free fields by these to approach the unstable, massless fix point, we reconstruct the massive continuum free field in the scaling limit.

More precisely, we invoke a formulation [8, 9] of the Wilson-Kadanoff renormalization group [10–12] in terms of operator algebras which is dual to the density matrix renormalization group (DMRG) [13–15]. We explicitly implement it in real space for scalar lattice fields in any dimension based. Inspired by renormalization in classical systems [16], we achieve this by a scaling function and its multiresolution analysis (MRA), cp. [17, 18]. Restricting to compactly supported wavelets, e.g. the Daubechies family, we obtain sharp localization with adjustable regularity compared with standard block-spin renormalization. This avoids certain obstacles encountered in [2, 4, 5], see also [19]. Pictorially, while the block-spin renormalization averages uniformly over adjacent sites, the wavelet renormalization takes a weighted average over the sites determined by the length of the low-pass filter associated with the scaling function. In the important case of harmonic (free) fields [13, 20] we control all involved objects. We point out that our approach yields a rigorous proof that spacetime locality (in the sense of the Haag-Kastler axioms [21]) in the continuum follows from Lieb-Robinson bounds [22–26]. For interacting lattice models, we do not expect to find all objects in closed form, but approximations by analytical and numerical expansion or perturbation methods will be required [15, 27, 28].

Real-space renormalization schemes received a rapidly growing interest in recent years, especially in the context of tensor networks [29] and the multi-scale entanglement renormalization ansatz (MERA) [30–32]. We show that our wavelet renormalization yields an analytic MERA [33, 34]. Mathematically, we identify the additional, discrete dimension of the $d+1$-dimensional tensor network determined by a MERA for a d-dimensional quantum system with the label set of an inductive system of operator algebras [35]. Each label represents a different scale of a given quantum system and the algebras consist of fields or observables at these scales. The connecting maps of the inductive system form the renormalization group acting on the dual state spaces by coarse graining.

OPERATOR-ALGEBRAIC RENORMALIZATION

For the renormalization scheme [9], we fix a family of lattices $\{\Lambda_N\}_{N\in\mathbb{N}_0}$ in $\mathbb{R}^d$ and consider a sequence of Hamiltonian quantum systems $\{\mathfrak{A}_N,\mathcal{H}_N,H_0^{(N)}\}_{N\in\mathbb{N}_0}$ indexed by the level N – the logarithmic scale accounting for the relative density of lattice points. At each level N, we have a concrete C^*-algebra $\mathfrak{A}_N\subset\mathcal{B}(\mathcal{H}_N)$ of basic field operators acting on a Hilbert space $\mathcal{H}_N$ and $H_0^{(N)}$ is a self-adjoint Hamiltonian on $\mathcal{B}(\mathcal{H}_N)$. Then, renormalization group theory considers (coarse graining) quantum operations, mapping a state on the finer system to one on the coarser system

$$\mathcal{E}_N^{N+M}(\rho_0^{(N+M)})=\rho_M^{(N)},\quad \mathcal{E}_N^{N+1}\circ\mathcal{E}_{N+1}^{N+1}=\mathcal{E}_N^{N+2},\qquad (1)$$

If these states are given by density matrices $\rho_0^{(N)}=(Z_0^{(N)})^{-1}e^{-H_0^{(N)}}$ and $\rho_M^{(N)}=(Z_M^{(N)})^{-1}e^{-H_M^{(N)}}$, the partition functions should be equal [36]: $Z_0^{(N+M)}=Z_M^{(N)}$. Generalizing from density matrices to algebraic states, (1) translates as:

$$\mathcal{E}_N^{N+M}(\omega_0^{(N+M)})=\omega_0^{(N+M)}\circ\alpha_{N+M}^N=\omega_M^{(N)},\qquad (2)$$

where $\alpha_{N+M}^N:\mathfrak{A}_N\to\mathfrak{A}_{N+M}$ is the dual of $\mathcal{E}_N^{N+M}$ (the ascending superoperators [31]). $\omega_0^{(N)}$ and $\omega_M^{(N)}$ are initial and renormalized states corresponding to $\rho_0^{(N)}$ and $\rho_M^{(N)}$. The equality between the state sums of the initial and renormalized states requires that α_{N+M}^N is unital and completely positive (ucp) sending states onto states and preserving probability [35]. We call the collection $\{\alpha^N_{N+M}\}_{M\in\mathbb{N}_0}$, the *scaling maps* or *renormalization group*. The semi-group property manifests as: $\alpha^{N+1}_{N+2}\circ\alpha^N_{N+1}=\alpha^N_{N+2}$. The structure is neatly summarized by Wilson's *triangle of renormalization* [11, p. 790] in Figure 1.

$$\begin{array}{ccccccccc}
 & & & & \iddots & & & \vdots & \vdots \\
 & & & & \omega_0^{(N+2)} \longrightarrow & & \cdots & \omega_\infty^{(N+2)} & \mathfrak{A}_{N+2} \\
 & & & & \big\downarrow \mathcal{E}^{N+2}_{N+1} & & & & \big\uparrow \alpha^{N+1}_{N+2} \\
 & & \omega_0^{(N+1)} \longrightarrow & & \omega_1^{(N+1)} \longrightarrow & & \cdots & \omega_\infty^{(N+1)} & \mathfrak{A}_{N+1} \\
 & & \big\downarrow \mathcal{E}^{N+1}_{N} & & \big\downarrow \mathcal{E}^{N+1}_{N} & & & & \big\uparrow \alpha^{N}_{N+1} \\
\omega_0^{(N)} & \longrightarrow & \omega_1^{(N)} & \longrightarrow & \omega_2^{(N)} & \longrightarrow & \cdots & \omega_\infty^{(N)} & \mathfrak{A}_N
\end{array}$$

FIG. 1: An analogue of Wilson's triangle of renormalization.

Now, given a sequence of initial states $\{\omega_0^{(N)}\}_{N\in\mathbb{N}_0}$, we define its *scaling limit* as the sequence $\{\omega_\infty^{(N)}\}_{N\in\mathbb{N}_0}$, where $\lim_{M\to\infty}\omega_M^{(N)}=\omega_\infty^{(N)}$ is the limit state on $\mathfrak{A}_N$ for fixed N. A scaling limit satisfies the consistency property (coarse-graining stability),

$$\omega_\infty^{(N')}\circ\alpha^N_{N'}=\omega_\infty^{(N)}, \qquad N<N', \tag{3}$$

whenever $\omega_\infty^{(N)}$ exists in the weak*-sense for every N. A scaling limit defines a state $\omega_\infty^{(\infty)}=\varprojlim_N\omega_\infty^{(N)}$ on the inductive limit algebra $\varinjlim_N\mathfrak{A}_N=\mathfrak{A}$, whenever it exists. Physically, non-trivial scaling limits require the divergence of correlation lengths $\xi_M^{(N)}$ (w.r.t. M) defined by $\omega_M^{(N)}$ between pairs of local operators. Thus, natural candidates for the initial states are ground states ω_{λ_N} of lattice Hamiltonians H_{λ_N} admitting quantum critical points $\lambda_N^{(c)}$ [37]. In this case, we can also consider the limit of dynamics, $\eta_t^{(N)}=e^{itH_0^{(N)}}(\,\cdot\,)e^{-itH_0^{(N)}}$, under refinement. More precisely, for fixed $t\in\mathbb{R}$ and $a_N\in\mathfrak{A}_N$, one asks the convergence of the sequence

$$\{\alpha_\infty^{N'}(\eta_t^{(N')}(\alpha^N_{N'}(a_N)))\}_{N'>N}, \tag{4}$$

in a suitable operator topology on $\mathfrak{A}$ relative to the scaling limit $\omega_\infty^{(\infty)}$ [38, 39]. Here, $\alpha_\infty^N:\mathfrak{A}_N\to\mathfrak{A}$ are the natural embeddings into inductive limit. This way we may define the limit $\eta_t^{(\infty)}=\lim_{N\to\infty}\eta_t^{(N)}$ and obtain a scaling-limit Hamiltonian $H_\infty^{(\infty)}$.

WAVELETS AND THE SCALAR FIELD

We now apply the above framework to lattice scalar fields, setting up a specific renormalization scheme involving compactly supported wavelets [7, 40]. A sequence of lattices $\Lambda_N=\varepsilon_N\{-r_N,...,0,...,r_N-1\}^d$ with scale parameters $\varepsilon_N=2^{-N}\varepsilon>0$, $r_N=2^Nr\in\mathbb{N}$ represents a discretization of the torus $[-L,L)^d=\mathbb{T}^d_L$ (the product $\varepsilon_Nr_N=L$ is fixed and we impose periodic boundary conditions: $r_N\equiv -r_N$). The kinematical setup of the lattice scalar field systems is given in terms of the one-particle spaces $\mathfrak{h}_N=\ell^2(\Lambda_N)$ [16, 41]: $\mathfrak{A}_N=\mathcal{W}(\mathfrak{h}_N)=\mathcal{W}_N$, $\mathcal{H}_N=\mathfrak{F}_+(\mathfrak{h}_N)\cong\bigotimes_{x\in\Lambda_N}\mathcal{H}_x$, where $\mathcal{H}_x=L^2(\mathbb{R})$, and $\mathcal{W}(\mathfrak{h}_N)=\mathcal{W}_N$ is the Weyl algebra,

$$W_N(\xi)W_N(\eta)=e^{-\frac{i}{2}\sigma_N(\xi,\eta)}W_N(\xi+\eta), \tag{5}$$

of $\mathfrak{h}_N$ w.r.t. the standard symplectic form, $\sigma_N=\Im\langle\cdot,\cdot\rangle_{\mathfrak{h}_N}$, and the decomposition into real subspaces facilitated by $\xi=\varepsilon_N^{\frac{d+1}{2}}q+i\varepsilon_N^{\frac{d-1}{2}}p$ for $\xi\in\mathfrak{h}_N$ using canonical scaling dimensions. We also need the dual lattices $\Gamma_N=\frac{\pi}{L}\{-r_N,...,0,...,r_N-1\}^d$ and the identification $\ell^2(\Lambda_N)\cong\ell^2(\Gamma_N,(2r_N)^{-d})$ via the discrete Fourier transform, $\mathcal{F}_N[\xi](k)=\sum_{x\in\Lambda_N}\xi(x)e^{ikx}=\hat{\xi}(k)$. For the real decomposition of $\ell^2(\Gamma_N,(2r_N)^{-d})$, we choose the normalization: $\hat{q}=\varepsilon_N^{\frac{d}{2}}\mathcal{F}_N[q]$, $\hat{p}=\varepsilon_N^{\frac{d}{2}}\mathcal{F}_N[p]$. Then, we choose symplectic maps $\{R^N_{N'}\}_{N\in\mathbb{N}_0}$ between one-particle spaces,

$$R^N_{N'}:\mathfrak{h}_N\to\mathfrak{h}_{N'},\quad \alpha^N_{N'}(W_N(\xi))=W_{N'}(R^N_{N'}(\xi)). \tag{6}$$

The renormalization group element $\alpha^N_{N'}$ is the second quantization of $R^N_{N'}$. The choice of the maps $R^N_{N'}$ is the most important step in our framework, and it determines the existence of the continuum scaling limit. Although there is an obvious sequence of inclusions associated with the inclusion $\Lambda_N\subset\Lambda_{N+1}$,

$$\mathfrak{h}_0...\subset\mathfrak{h}_N\subset\mathfrak{h}_{N+1}\subset..., \tag{7}$$

we do *not* take $R^N_{N'}$ as these inclusion maps. Instead, we use the *scaling equation* [7, 42, 43]:

$$\phi(x)=\sum_{n\in\mathbb{Z}^d}h_n2^{\frac{d}{2}}\phi(2x-n), \tag{8}$$

where ϕ is a scaling function, s.t. $\{\phi(\,.\,-n)\}_{n\in\mathbb{Z}^d}$ is orthonormal. As we intend to built local operators, we specialize to a compactly supported scaling function ϕ normalized by $\hat{\phi}(0)=1$. Such a ϕ generates an orthonormal, compactly supported wavelet basis in $L^2(\mathbb{R}^d)$, and the sum (8) is necessarily finite (h_n is a finite low-pass filter [7]). We denote by $\phi_x^{(\varepsilon)}(\,\cdot\,)=\varepsilon^{-\frac{d}{2}}\phi(\varepsilon^{-1}(\,\cdot\,-x))$ the scaling function localized near $x\in\varepsilon\mathbb{Z}^d$ at length scale ε. The orthonormality property of the scaling function, $(\phi_x^{(\varepsilon)},\phi_y^{(\varepsilon)})_{L^2(\mathbb{R}^d)}=\delta_{x,y}$, $x,y\in\varepsilon\mathbb{Z}^d$, allows us to identify the linear span of $\{\phi_x^{(\varepsilon)}\}$ with $\ell^2(\varepsilon\mathbb{Z}^d)$. Periodizing $\{\phi_{n\varepsilon}^{(\varepsilon)}\}_{n\in\mathbb{Z}^d}$ on the torus $\mathbb{T}^d_L$, we formally identify, $\delta_x^{(0)}\sim\phi_x^{(\varepsilon)}$, $x\in\Lambda_0$, with the standard basis of $\mathfrak{h}_0=\ell^2(\Lambda_0)$. With this identification and the scaling relation (8) in mind, we define R^N_{N+1}:

$$R^N_{N+1}(\delta_x^{(N)})=2^{\frac{d}{2}}\sum_{n\in\mathbb{Z}^d}h_n\delta^{(N+1)}_{x+n\varepsilon_{N+1}}, \tag{9}$$

where $\{\delta_x^{(N)}\}_{x\in\Lambda_N}$ is the standard basis of $\ell^2(\Lambda_N)$. Choosing $\log_2 r\in\mathbb{N}_0$ ensures completeness of (7) in $L^2(\mathbb{T}^d_L)$.

The properties of the scaling functions entail that R^N_{N+1} is symplectic. From (9), we derive the asymptotic map:

$$R^N_\infty(q,p) = \sum_{x\in\Lambda_N} (q,p)(x)\phi(\varepsilon_N^{-1}(\cdot - x)). \quad (10)$$

By (6), we have an inductive system of Weyl algebras determining an inductive-limit Weyl algebra [35]: $\varinjlim_N \mathcal{W}_N = \mathcal{W}$. By the Gelfand-Naimark-Segal (GNS) construction with respect to a scaling limit $\omega^{(\infty)}_\infty$ of a suitable sequence of initial states, $\mathcal{W}$ embeds in the continuum field theory, as we see below. In accordance with [2, 3, 8, 9, 44] we call $\mathcal{W}$ the *semi-continuum limit*, see also [45, 46].

Connection with multi-scale entanglement renormalization

We can use the structure (9) of the map R^N_{N+1} to exhibit a connection with the MERA [31, 32]. We decompose R^N_{N+1} into the trivial isometry I^N_{N+1} resulting from the inclusion $\Lambda_N \subset \Lambda_{N+1}$ and the symplectic rotation S_{N+1} with kernel:

$$S_{N+1}(x-y) = \sum_{n\in\mathbb{Z}^d} h_n \delta_{n,\varepsilon^{-1}_{N+1}(x-y)}, \quad x,y\in\Lambda_{N+1}. \quad (11)$$

Then, the renormalization group element α^N_{N+1} takes the form expected from a MERA [9, 30, 44]:

$$\alpha^N_{N+1}(\cdot) = U_{N+1}(\cdot\otimes\mathbb{1}_{N+1\setminus N})U^*_{N+1}, \quad (12)$$

where $\mathbb{1}_{N+1\setminus N}$ is the identity on the ancillary Fock space $\mathfrak{F}_+(\Lambda_{N+1}\setminus\Lambda_N)$ and U_{N+1} is the Bogoliubov unitary induced by S_{N+1}. The dual quantum channel $\mathcal{E}^{N+1}_N = \mathrm{Tr}_{N+1\setminus N}(U^*_{N+1}(\cdot)U_{N+1})$ is given by the twisted partial trace on the ancillary. The actual MERA isometry and disentangler are recovered in combined form by the GNS isometry V^N_{N+1} induced by α^N_{N+1} due to (3) in the GNS representation of a scaling limit $\omega^{(\infty)}_\infty$:

$$\Omega^{(N+1)}_\infty = V^N_{N+1}\Omega^{(N)}_\infty, \qquad \varinjlim_{N} \Omega^{(N)}_\infty = \Omega^{(\infty)}_\infty \quad (13)$$

where $\Omega^{(N)}_\infty$ is the vector inducing $\omega^{(N)}_\infty$ on $\mathcal{W}_N$. In this sense, even our general renormalization group scheme produces an operator-algebraic MERA in the form of the maps $\alpha^N_{N'}$ and a scaling limit. Therefore, the scaling limits of free lattice ground states on $\mathcal{W}_N$, which we construct below, exhibit the structure of an analytic MERA [33, 47].

Spatial locality structure and translations

One of the most important properties of the renormalization group elements $\alpha^N_{N'}$ defined by (9) is their spatial localization which is encoded into the low-pass filter h_n. Defining the spatial support $\mathrm{supp}(a_N)$ of a local operator a_N at level N as the set of sites $x\in\Lambda_N$ s.t. the restriction $a_{N|x}\neq\mathbb{1}$ is compatible with the notion of support of elements in the one-particle space $\mathfrak{h}_N$. From (9) and the linearity (unitality) of $R^N_{N'}$ ($\alpha^N_{N'}$) it follows that the increase in support due to renormalization group steps is bounded from above by $2^{-N}(r_{\min}-1)$, where $r_{\min}$ is the number of nonzero h_n's. Thus, we can define local algebras $\mathcal{W}(\mathcal{S})\subset\mathcal{W}$ for open sets $\mathcal{S}\subset\mathbb{T}^d_L$ by collecting at each level N all the operators a_N with support in the sublattice $\Lambda_N(\mathcal{S}) = \Lambda_N\cap\mathcal{S}$ of the points $x\in\Lambda_N\cap\mathcal{S}$ s.t. the cube $x+[0,\varepsilon_N(r_{\min}-1)]^d$ does not intersect the boundary $\partial\mathcal{S}$. The bound on the increase of support ensures that this definition is stable under the renormalization group elements $\alpha^N_{N'}$. In other words, we have a local inductive limit: $\mathcal{W}(\mathcal{S}) := \varinjlim_N \mathcal{W}_N(\mathcal{S}) = \varinjlim_N \mathcal{W}(\mathfrak{h}_N(\mathcal{S}))$, with $\mathfrak{h}_N(\mathcal{S}) = \ell^2(\Lambda_N(\mathcal{S}))$. Then $\mathcal{W} = \overline{\bigcup_{\mathcal{S}}\mathcal{W}(\mathcal{S})}$ is a quasi-local algebra [38] because:

$$\mathcal{W}(\mathcal{S})\subset\mathcal{W}(\mathcal{S}') \qquad \mathcal{S}\subset\mathcal{S}', \quad (14)$$

$$[\mathcal{W}(\mathcal{S}),\mathcal{W}(\mathcal{S}')] = \{0\} \qquad \mathcal{S}\cap\mathcal{S}' = \emptyset. \quad (15)$$

As the semi-continuum limit $\mathcal{W}$ is an algebra associated with the Cantor set Λ_∞ of the dyadic rationals scaled by ε which results from the refinement of Λ_N in the limit $N\to\infty$, there is a natural action by translations ρ which brings $\mathcal{W}(\mathcal{S})$ to $\mathcal{W}(\mathcal{S}+x)$ for $x\in\Lambda_\infty$ [48]. Whether this extends to a continuous action of $\mathbb{R}^d$ in the scaling limit depends on the choice of the initial states $\{\omega^{(N)}_0\}_{N\in\mathbb{N}_0}$, cp. [2, 4, 9]. In $d=1$, $\mathcal{W}$ even admits a representations of Thompson's groups by identifying Λ_N with a complete binary tree of depth N [48].

SCALING LIMITS OF HARMONIC LATTICE SYSTEMS

We are now in a position to apply the renormalization group $\{\alpha^N_{N'}\}_{N<N'}$ defined by (9) to find the ground-state scaling limits of the free lattice Hamiltonian on $\mathcal{H}_N$:

$$H^{(N)}_0 = \frac{\varepsilon_N^{-1}}{2}\left(\sum_{x\in\Lambda_N}\Pi^2_{N|x} + \mu^2_N\Phi^2_{N|x} - 2\sum_{\langle x,y\rangle\subset\Lambda_N}\Phi_{N|x}\Phi_{N|y}\right) \quad (16)$$

where $\mu_N\geq 2d$ is a "mass" parameter and Φ_N,Π_N are the dimensionless canonical field and momentum operators at level N identified via $W_N(\xi) = e^{i(\Phi_N(\Re\xi)+\Pi_N(\Im\xi))}$. The ground state $\Omega_{\mu_N,0}$ of $H^{(N)}_0$ can be encoded into the state:

$$\omega_{\mu_N,0}(W_N(\xi)) = e^{-\frac{1}{4}\left|\left|\gamma^{-1/2}_{\mu_N}\hat{q}_N + i\gamma^{1/2}_{\mu_N}\hat{p}_N\right|\right|^2_{\mathfrak{h}_N}}, \quad (17)$$

with the dispersion relation $\gamma^2_{\mu_N}(k) = \varepsilon_N^{-2}(\mu^2_N - 2d) + 2\epsilon_N^{-2}\sum_{j=1}^d(1-\cos(\epsilon_N k_j))$, $k\in\Gamma_N$. The GNS construction applied to $\mathcal{W}_N$ w.r.t. $\omega_{\mu_N,0}$ yields a representation which is unitarily equivalent to that on $\mathcal{H}_N$ s.t. $\Omega_{\mu_N,0}$ is identified with the cyclic GNS vector.

Scaling limit of the ground states

Let us now explain how ground-state scaling limits of (16) can be constructed in any spatial dimension $d \geq 1$: Choosing (17) for every N as our initial family of states, we generate a sequence of states $\{\omega_M^{(N)}\}_{M\in\mathbb{N}_0}$ at each level N (Figure 1). To avoid the fix points $\mu_N^2 = 2d$ (massless, unstable) and $\mu_N^2 = \infty$ (ultralocal, stable) of the renormalization group and hit the unstable manifold of the relevant Φ^2-operator, we need to satisfy the *renormalization condition*,

$$\lim_{N\to\infty} \varepsilon_N^{-2}(\mu_N^2 - 2d) = m^2, \tag{18}$$

for some $m > 0$. Then, the massive continuum dispersion relation results from $\lim_{M\to\infty} \gamma_{\mu_{N+M}}(k)^2 = m^2 + k^2 = \gamma_m(k)^2$ and the scaling limit is (using (9) & (17)):

$$\omega_{m,\infty}^{(N)}(W_N(\xi)) = e^{-\frac{1}{4}\left\|\hat{\phi}_0^{(\varepsilon_N)}(\gamma_m^{-1/2}\hat{q}+i\gamma_m^{1/2}\hat{p})\right\|_{2,L}^2}. \tag{19}$$

Here, $\hat{q}, \hat{p}$ are periodically extended to $\frac{\pi}{L}\mathbb{Z}^d$, and $\|\cdot\|_{2,L}$ is the standard norm of $\ell^2(\frac{\pi}{L}\mathbb{Z}^d, (2L)^{-d})$. Since the contribution to $\omega_{m,\infty}^{(N)}$ involving $\hat{p}$ is the most singular, the limit states are well-defined for scaling functions with momentum-space decay $|\hat{\phi}(k)| \leq C(1+|k|)^{-\rho}$ s.t. $\rho > \frac{d+1}{2}$. This condition can be satisfied, for example, by a scaling function associated with the Daubechies (D2K) wavelet family, $\{{}_K\phi\}_{K\in\mathbb{N}}$ [7]. A sufficient choice for (19) is a tensor product of D4 wavelets, $\phi = {}_2\phi^{\otimes d}$, because $\rho \approx 1.339$ for this choice. The formula (19) for $\omega_{m,\infty}^{(N)}$ equals the evaluation of the usual continuum ground state $\omega_m^{(L)}$ of mass m for finite volume L on the continuum Weyl operators $W(f,g) = e^{i(\Phi(f)+\Pi(g))}$ with $(f,g) = R_\infty^N(q,p)$. Therefore, $\omega_{m,\infty}^{(\infty)}$ agrees with $\omega_m^{(L)}$ on the span of wavelets associated with ϕ. But, because of the localization of these wavelets, their density in the usual Sobolev spaces $H^s(\mathbb{R}^d)$ for sufficiently high regularity [7, 40] and the strong continuity of Weyl operators in the GNS representation $\pi_{m,\infty}^{(\infty)}$ [41], we conclude that the local von Neumann algebras $\pi_{m,\infty}^{(\infty)}(\mathcal{W}(\mathcal{S}))''$ equal those of the continuum free field $\mathcal{A}_{m,L}(\mathcal{S})$[49] in finite volume. Since the scaling limit $\omega_{m,\infty}^{(\infty)}$ is evidently invariant w.r.t. dyadic spatial translations, ρ_x is implemented by a unitary $V_x^{(\infty)}$. These unitaries can be extended to $x \in \mathbb{T}_L^d$ by continuity in the strong operator topology because they coincide with the translations in the continuum for dyadic x, and the momentum operators can be defined. The thermodynamical limit of (19), $L \to \infty$, exists by a Riemann-sum argument and yields the free, massive vacuum ω_m in infinite volume together with its local time-zero algebras $\mathcal{A}_m(\mathcal{S})$.

Dynamics, locality and Lieb-Robinson bounds

The dynamics $\eta^{(N)} : \mathbb{R} \curvearrowright \mathcal{W}_N$ of the free lattice Hamiltonian $H_0^{(N)}$ is given by the harmonic time-evolution, $\tau^{(N)} : \mathbb{R} \curvearrowright \mathfrak{h}_N$, on the one-particle space using γ_{μ_N}. Then, $\eta_t^{(N)}(W_N(\hat{q},\hat{p})) = W_N(\tau_t^{(N)}(\hat{q},\hat{p}))$, is well-defined because $\tau_t^{(N)}$ is symplectic. The initial states are preserved by the dynamics, i.e. $\omega_{\mu_N,0} \circ \eta_t^{(N)} = \omega_{\mu_N,0}$. As explained in the general context, we understand the convergence of the lattice dynamics to a dynamics for the scaling limit via sequences (4) with $a_N = W_N(q,p)$. α_∞^N is explicitly realized by (10). As a consequence of (8), we have $R_\infty^{N'} \circ R_{N'}^N = R_\infty^N$ for $N < N'$ with analogous identities for α_∞^N. Since γ_{μ_N} extends periodically to $\Gamma_{N'}$ for $N < N' \leq \infty$, we find:

$$\lim_{N'\to\infty} \|R_\infty^{N'}(\tau_t^{(N')}(R_{N'}^N(\hat{q},\hat{p}))) - \tau_t^{(\infty)}(R_\infty^N(\hat{q},\hat{p}))\| = 0, \tag{20}$$

for all N w.r.t. the closure $\overline{\mathfrak{h}}_\infty$ in the norm $\|\cdot\|$ defined by (19). $\tau^{(\infty)} : \mathbb{R} \curvearrowright \overline{\mathfrak{h}}_\infty$ is defined similarly to $\tau^{(N)}$ using γ_m. As $\gamma_{\mu_{N'}} \to \gamma_m$ we deduce

$$\lim_{N'\to\infty} \pi_{m,\infty}^{(\infty)} \circ \alpha_\infty^{N'} \circ \eta_t^{(N')} \circ \alpha_{N'}^N = \eta_t^{(\infty)} \circ \pi_{m,\infty}^{(\infty)} \circ \alpha_\infty^N, \tag{21}$$

pointwise strongly on each $\mathfrak{A}_N$ and uniformly on bounded intervals of $t \in \mathbb{R}$. By construction, $\eta^{(\infty)} : \mathbb{R} \curvearrowright \pi_{m,\infty}^{(\infty)}(\mathcal{W})''$ is the time evolution of the continuum free scalar field commuting with ρ. It is implemented by a unitary group $U_t^{(\infty)} = e^{itH_\infty^{(\infty)}}$ with the (renormalized) free continuum Hamiltonian $H_\infty^{(\infty)}$ as its generator because the scaling limit $\omega_{m,\infty}^{(\infty)}$ is invariant under $\eta^{(\infty)}$. Explicitly, $H_\infty^{(\infty)}$ is the second quantization of the generator $h_\infty^{(\infty)}$ of $\tau^{(\infty)}$ on its natural domain [50]. Identifying $\overline{\mathfrak{h}}_\infty$ with $\ell^2(\frac{\pi}{L}\mathbb{Z}^d, (2L)^{-d})$, $h_\infty^{(\infty)}$ is given by (right) multiplication with the matrix-valued function $i\gamma_m\sigma_2$ (with σ_2 the second Pauli matrix). Since γ_m is the free, massive relativistic dispersion relation, we know that $\eta^{(\infty)}$ has propagation speed $c = 1$ and, thus, obtain a causal net of local von Neumann algebras for suitable $\mathcal{O} \subset \mathbb{R} \times \mathbb{T}_L^d$ [1, 21]:

$$\mathcal{A}_{m,L}(\mathcal{O}) = \left(\bigcup_{t\in\mathbb{R}} \eta_t^{(\infty)}(\pi_{m,\infty}^{(\infty)}(\mathcal{W}(\mathcal{O}(t)))'')\right)'', \tag{22}$$

where $\mathcal{O}(t) = \{x \,|\, (t,x) \in \mathcal{O}\} \subset \mathbb{T}_L^d$. A more lattice-intrinsic and model-independent way to conclude that (22) defines a causal net is via Lieb-Robinson bounds [23, 24]. Considering the periodic extension of $\eta^{(N)}$ to $\mathcal{W}$, e.g. by (12), said bounds for harmonic lattice systems [25] imply:

$$\lim_{N\to\infty} \left\| \left[\eta_t^{(N)}(\mathcal{W}(\mathcal{S})), \mathcal{W}(\mathcal{S}')\right] \right\| = 0, \tag{23}$$

exponentially fast and uniformly for $|t| \leq T$ with $\mathcal{S}_{c'T} = \{x \,|\, \mathrm{dist}(x,\mathcal{S}) \leq c'T\}$ and $\mathcal{S}' \cap \mathcal{S}_{c'T} = \emptyset$ for some $c' > 1$. Because $c' > 1$, the causality implied by (23) is not strict, but that is rather not due to the argument than a non-optimal bound on the Lieb-Robinson velocity [23].

CONCLUSIONS AND OUTLOOK

Our results show that the existence of continuum limits depends decisively on the choice of a renormalization scheme. Using compactly supported wavelets and correctly choosing the initial states allows us to reconstruct the continuum field theory from the lattice approximation through the semi-continuum limit.

This new bridge between lattice and continuum field theory may help to investigate other problems in quantum field theory. As our general method can include fermions, we expect that an application of the wavelet method to (free) lattice fermions leads to similar results as those presented here (adjusting the one-particle scaling maps to be unitary). In this respect, it would be very interesting to exhibit precise relations with other renormalization schemes using wavelets [33, 47]. Since we are able to construct the complete renormalization group trajectory for the Φ^2-operator, a rigorous proof of a (restricted) c-theorem [51] is conceivable using the concept of entanglement entropy [52, 53]. In $d = 1$, Jones' construction yields a geometric representation of Thompson's group T on $\mathcal{W}$ interpretable as an action of discretized, orientation-preserving diffeomorphisms of the circle [48]. But, expecting a genuine extension to diffeomorphisms in the continuum limit to obtain a CFT, similar to the translations, is probably too naive [54]. Continuing along these lines, although our methods should be able to construct the $\mathrm{U}(1)$-current from the lattice ($\mu_N^2 = 2d$), the induced Jones action by T corresponds to diffeomorphisms acting on the time-zero slices and not along light rays. The wavelet method works in principle also for interacting lattice systems and the existence of scaling limits is ensured by weak*-compactness of the state spaces. Moreover, Lieb-Robinson bounds for anharmonic lattice systems [26] offer a possibility to obtain spacetime locality directly from the lattice [23, 24]. Parts of the classical results by Glimm-Jaffe and others [1] on interacting models in $d=1$ can be formulated in terms of our method using a low-pass filter that implements momentum-space cutoffs [6] and should extend to the wavelet setting. But, those results indicate as well that proving convergence to the scaling limit is difficult [55].

Helpful discussions with T. Osborne, A. Abdesselam and M. Fröb are acknowledged by AS. VM and GM are partially supported by the European Research Council Advanced Grant 669240 QUEST. AS is supported by the Humboldt Foundation through a Feodor Lynen Return Fellowship. VM is titolare di un Assegno di Ricerca dell'Istituto Nazionale di Alta Matematica (INdAM fellowship). YT is supported by Programma per giovani ricercatori, anno 2014 "Rita Levi Montalcini" of the Italian Ministry of Education, University and Research. VM, GM and YT also acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome "Tor Vergata", CUP E83C18000100006 and the University of Rome "Tor Vergata" funding scheme "Beyond Borders", CUP E84I19002200005.

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