The Bisognano-Wichmann property for asymptotically complete massless QFT

Wojciech Dybalski

Zentrum Mathematik, Technische Universität München,
E-mail: `dybalski@ma.tum.de`

Vincenzo Morinelli

Dipartimento di Matematica, Università di Roma "Tor Vergata"
E-mail: `morinell@mat.uniroma2.it`

Abstract

We prove the Bisognano-Wichmann property for asymptotically complete Haag-Kastler theories of massless particles. These particles should either be scalar or appear as a direct sum of two opposite integer helicities, thus, e.g., photons are covered. The argument relies on a *modularity condition* formulated recently by one of us (VM) and on the Buchholz' scattering theory of massless particles.

1 Introduction

For any von Neumann algebra with a faithful state Tomita-Takesaki theory gives a natural dynamics constructed using the complex structure of the algebra. It was shown by Bisognano and Wichmann that for the algebra of the Wightman fields localised in a spacelike wedge this latter dynamics coincides with the Lorentz boosts in the direction of this wedge [BW76]. Furthermore, in the presence of the Bisognano-Wichmann (B-W) property the full global symmetry of the model is contained in the modular structure of the net reducing the dichotomy between symmetries and algebras to an inclusion [BGL95]. The B-W property is also important for many other reasons, ranging from the intrinsic meaning of the CPT symmetry [GL95] to a construction of interacting models [Le08] and to entanglement theory [W18]. While its formulation is most natural in the algebraic (Haag-Kastler) setting, and it is known to hold in all the 'physical' examples, its general proof in this framework is missing to date. The reason is the broadness of the Haag-Kastler setting which admits also non-physical counterexamples to the B-W property. For example, when an infinite family of massive spinorial or infinite spin particles occurs [LMR16, Mo18]. Thus it is important to find natural assumptions which exclude such pathological cases.

For massless theories the assumption of global conformal invariance implies the B-W property as shown in [BGL93]. A search for an algebraic sufficient condition for

the B-W property, not relying on conformal covariance, was started in [Mo18, Mo17] at the level of one particle nets. Here a criterion on the covariant representation called the *modularity condition* was shown to give the B-W property of the one particle net. For massive theories which are asymptotically complete, the paper by Mund [Mu01] gives the B-W property. This paper exploits a result of Buchholz and Epstein [BE85] in order to study geometrically the analytic extension of one parameter boosts and identify it with the associated modular operator. This allows to verify the Bisognano-Wichmann property on the one particle subspace and conclude it for the full interacting net by asymptotic completeness and wedge localization of the modular operator. Unfortunately this method does not apply to massless theories as the argument of Buchholz and Epstein requires a mass gap.

In the present paper we prove the B-W property for massless bosonic theories which are asymptotically complete, by combining some ideas contained in the works mentioned above. First, we identify the modular operator and the boost generator associated to the same wedge at the single-particle level. To this end, we verify the modularity condition introduced in [Mo17, Mo18] and thus avoid the use of the Buchholz-Epstein result. Our argument requires that the representation of the Poincaré group is either scalar or a direct sum of two representations with opposite integer helicities. Thereby we show that the modularity condition applies to a large family of massless representations, including higher helicity. Next, we show that the B-W property holds on the entire Hilbert space by using scattering theory and the assumption of asymptotic completeness. We recall that scattering theory for massless bosons was developed in [Bu77] and various simplifications have been found meanwhile. In the present paper we use the variant from [AD17] which is based on novel ergodic theorem arguments and on uniform energy bounds on asymptotic fields from [Bu90, He14.1]. Our results extend the range of validity of the B-W property and reconfirm its status as a generic property of physically reasonable models.

Our paper is organized as follows: In Sect. 2 we state our main result after the necessary preparations. In Sect. 3 we recall some relevant facts from scattering theory of massless particles, the theory of standard subspaces and one-particle nets, and representations of the Poincaré group. In Sect. 4 the modularity condition is stated and verified for one-particle massless nets with arbitrary integer spin. In Sect. 5 the result is generalised to an arbitrary number of particles using scattering theory and the assumption of asymptotic completeness.

2 Framework and results

2.1 Local nets and the Bisognano-Wichmann property

Let $\mathbb{R}^{1+3}$ be the Minkowski spacetime. We denote by $\mathcal{K}$ the family of double cones $\mathcal{O} \subset \mathbb{R}^{1+3}$ ordered by inclusion and write $\mathcal{O}'$ for the causal complement of $\mathcal{O}$ in $\mathbb{R}^{1+3}$. Furthermore, let $\widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes \mathrm{SL}(2,\mathbb{C})$ denote the covering group of the proper ortochronous Poincaré group $\mathcal{P}_+^\uparrow$. We denote with $\Lambda : \widetilde{\mathcal{P}}_+^\uparrow \to \mathcal{P}_+^\uparrow$ the covering map.

Definition 2.1. *Let $\mathfrak{H}$ be a fixed Hilbert space. We say that $\mathcal{K} \ni \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset B(\mathfrak{H})$ is a local net of von Neumann algebras in a vacuum representation if the following properties hold:*

1. ***Isotony:** $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$.*

2. ***Poincaré covariance:** there is a continuous unitary representation U of $\widetilde{\mathcal{P}}_+^\uparrow$ such that*

$$U(\lambda)\mathcal{A}(\mathcal{O})U(\lambda)^* = \mathcal{A}(\lambda\mathcal{O}) \quad \text{for} \quad \lambda \in \widetilde{\mathcal{P}}_+^\uparrow. \tag{2.1}$$

3. ***Positivity of the energy:** the joint spectrum of translations in U is contained in the forward lightcone $V_+ = \{p \in \mathbb{R}^{1+3} : p^0 \geq 0, \, p^2 = (p,p) \geq 0\}$.*

4. ***Cyclicity of the vacuum:** there is a unique (up to a phase) unit vector $\Omega \in \mathfrak{H}$, the physical vacuum state, which is U-invariant and cyclic for the global algebra $\mathcal{A} := \overline{\bigcup_{\mathcal{O} \subset \mathbb{R}^{1+3}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}$ of the net.*

5. ***Locality:** $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)'$ for $\mathcal{O}_1 \subset \mathcal{O}_2'$.*

A local net of von Neumann algebras will be denoted by $(\mathcal{A}, U, \Omega)$.

For future reference, we set for any region $\mathcal{U} \subset \mathbb{R}^{1+3}$

$$\mathcal{A}_{\mathrm{loc}}(\mathcal{U}) := \bigcup_{\mathcal{O} \subset \mathcal{U}} \mathcal{A}(\mathcal{O}) \quad \text{and} \quad \mathcal{A}(\mathcal{U}) := \mathcal{A}_{\mathrm{loc}}(\mathcal{U})'' \tag{2.2}$$

and we refer to $\mathcal{A}_{\mathrm{loc}} := \mathcal{A}_{\mathrm{loc}}(\mathbb{R}^{1+3})$ as the algebra of strictly local operators.

In order to introduce the B-W property, we need some geometric preliminaries: a **wedge shaped region** $W \subset \mathbb{R}^{1+3}$ is an open region of the form gW_1 where $g \in \mathcal{P}_+^\uparrow$ and $W_1 = \{x \in \mathbb{R}^{1+3} : |x_0| < x_1\}$. The set of wedges is denoted by $\mathcal{W}$. Note that if $W \in \mathcal{W}$, then $W' \in \mathcal{W}$, where prime denotes here the spacelike complement. It is possible to associate to any wedge a one-parameter group of boosts Λ_W fixing the wedge W by the following formula for W_1

$$\mathbb{R} \ni t \to \Lambda_{W_1}(t) := \begin{pmatrix} \cosh(t) & \sinh(t) & 0 & 0 \\ \sinh(t) & \cosh(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

and the covariant action of the Poincaré group on the set of wedges.

We call $W_\alpha = \{x \in \mathbb{R}^{1+3} : |x_0| < x_\alpha\}$, $\alpha = 1, 2, 3$, the wedge in the x_α direction and with R_α, Λ_α are the one-parameter groups of rotations and boosts, respectively, fixing W_α. Their unique one parameter group lifts to $\mathrm{SL}(2, \mathbb{C})$ are denoted r_α and λ_α. In general λ_W will denote the one parameter group lift of Λ_W. Note that $\lambda_\alpha(t) = e^{\frac{t}{2}\sigma_\alpha}$ and $r_\alpha(\theta) = e^{i\frac{\theta}{2}\sigma_\alpha}$ where $t, \theta \in \mathbb{R}$ and σ_α are the Pauli matrices. In particular one has that $r_\alpha(2\pi) = -I =: r(2\pi)$.

For any $W \in \mathcal{W}$ we define $\mathcal{A}(W)$ according to (2.2). It is well known that the vacuum is cyclic and separating for $\mathcal{A}(W)$ thus the Tomita-Takesaki theory gives the corresponding modular evolution $\mathbb{R} \ni t \mapsto \Delta_W^{it}$.

Definition 2.2. *We say that a local net $(\mathcal{A}, U, \Omega)$ satisfies the **Bisognano-Wichmann property** if for all $W \in \mathcal{W}$, $t \in \mathbb{R}$,*

$$U(\lambda_W(2\pi t)) = \Delta_W^{-it}.$$

2.2 Massless Wigner particles and asymptotic nets

Scattering theory of massless Wigner particles was developed by Buchholz [Bu77, Bu75], both in the bosonic and fermionic case. Recently the bosonic case was simplified in [AD17]. We collect below the main results in this subject following [AD17]. We first introduce the single-particle subspace.

Definition 2.3. *A local net $(\mathcal{A}, U, \Omega)$ describes massless Wigner particles if $\mathfrak{H}$ contains a subspace $\mathfrak{H}^{(1)} \neq \{0\}$ s.t.*

$$\mathrm{Ran}\mathbf{1}_{\{0\}}(M) = \mathbb{C}\Omega \oplus \mathfrak{H}^{(1)}, \tag{2.4}$$

where $\mathbf{1}_{\{0\}}(M)$ denotes the spectral projection of the mass operator $M := \sqrt{(P^0)^2 - \boldsymbol{P}^2}$ corresponding to the eigenvalue zero. We say that these particles have helicities $h_1, h_2, h_3 \ldots \in \mathbb{Z}$ if $U|_{\mathfrak{H}^{(1)}}$ is a finite or infinite multiple of the direct sum of the corresponding zero mass representations.

It is well known that to any local theory containing massless particles one can associate an asymptotic (free) theory [Bu77]. We outline now this construction following [AD17]. For the unitary representation of translations $U|_{\mathbb{R}^4}$ we shall write $U(x) = e^{i(P^0 x^0 - \boldsymbol{P} \cdot \boldsymbol{x})}$ and for translates of observables $A \in \mathcal{A}$ the notations $\alpha_x(A) := A(x) := U(x)AU(x)^*$ are used. If $g \in L^1(\mathbb{R}^4)$, then $A(g) := \int A(x)g(x)d^4x$ denotes the operator A smeared with the function g. Moreover, we set

$$\mathcal{A}_{\mathrm{loc},0} := \{A \in \mathcal{A}_{\mathrm{loc}} : x \mapsto A(x) \text{ smooth in norm}\}. \tag{2.5}$$

This is a weakly dense $*$-subalgebra of $\mathcal{A}_{\mathrm{loc}}$, as can be seen by smearing local operators with delta-approximating functions. Next, we specify the following Poincaré invariant subset of $C_0^\infty(\mathbb{R}^4)$

$$C_*(\mathbb{R}^4) := \{(n_\mu \partial^\mu)^5 g : g \in C_0^\infty(\mathbb{R}^4), \ n_0 = \sqrt{1 + \boldsymbol{n}^2}\} \tag{2.6}$$

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and define

$$\mathcal{A}_{C_*} := \{\, B(g) \,:\, B \in \mathcal{A}_{\mathrm{loc},0}, \ g \in C_*(\mathbb{R}^4) \,\}, \tag{2.7}$$

$$\mathcal{A}^{C_*} := \mathrm{Span}\,\mathcal{A}_{C_*}, \tag{2.8}$$

$$\mathcal{A}_{C_*}(\mathcal{O}) := \mathcal{A}_{C_*} \cap \mathcal{A}(\mathcal{O}), \quad \mathcal{A}^{C_*}(\mathcal{O}) := \mathcal{A}^{C_*} \cap \mathcal{A}(\mathcal{O}), \quad \mathcal{O} \in \mathcal{K}. \tag{2.9}$$

Now we move on to the construction of asymptotic fields of massless particles. For any $A \in \mathcal{A}^{C_*}$ and $f \in C^\infty(S^2)$, we set as in [Bu77, Bu82]

$$A_t\{f\} := -2\,t \int d\omega(\boldsymbol{n})\, f(\boldsymbol{n})\, \partial_0 A(t, t\boldsymbol{n}). \tag{2.10}$$

Here $d\omega(\boldsymbol{n}) = \frac{\sin\nu\, d\nu d\varphi}{4\pi}$ is the normalized, invariant measure on S^2 and $\partial_0 A := \partial_s(\mathrm{e}^{\mathrm{i}sH} A \mathrm{e}^{-\mathrm{i}sH})|_{s=0}$. In order to improve the convergence in the limit of large t, we proceed to time averages of $A_t\{f\}$, namely

$$\bar{A}_t\{f\} := \int dt'\, h_t(t')\, A_{t'}\{f\}. \tag{2.11}$$

Here for non-negative $h \in C_0^\infty(\mathbb{R})$, supported in the interval $[-1, 1]$ and normalized so that $\int dt\, h(t) = 1$, we set $h_t(t') = t^{-\bar{\varepsilon}} h(t^{-\bar{\varepsilon}}(t' - t))$ with $t \geq 1$ and $0 < \bar{\varepsilon} < 1$. It turns out that these limits exist on all vectors from the domain

$$D_{P^0} := \bigcap_{n \geq 1} D((P^0)^n), \tag{2.12}$$

where $D((P^0)^n)$ is the domain of self-adjointness of $(P^0)^n$.

Lemma 2.4. *Let $A \in \mathcal{A}^{C_*}(\mathcal{O})$ and $f \in C^\infty(S^2)$. Then, the limit*

$$A^{\mathrm{out}}\{f\}\Psi = \lim_{t \to \infty} \bar{A}_t\{f\}\Psi \tag{2.13}$$

exists for $\Psi \in D_{P^0}$ and is again an element of D_{P^0}.

The operators $A^{\mathrm{out}}\{f\}$ are constructed in such a way that they create single-particle states from the vacuum, namely

$$A^{\mathrm{out}}\{f\}\Omega = P^{(1)} f\left(\frac{\boldsymbol{P}}{|\boldsymbol{P}|}\right) A\Omega, \tag{2.14}$$

where $P^{(1)}$ is the projection on the single-particle subspace $\mathfrak{H}^{(1)}$. Vectors of the form (2.14) span a dense subspace of $\mathfrak{H}^{(1)}$, even in the case $f \equiv 1$. Furthermore, if $A^{\mathrm{out}}\{f\}$, $A'^{\mathrm{out}}\{f'\}$ are two asymptotic fields as specified above, then

$$[A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}] = \langle \Omega, [A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}]\Omega\rangle 1_{\mathfrak{H}} \tag{2.15}$$

as operators on D_{P^0}. For $f \equiv 1$ the operators $A^{\mathrm{out}}\{f\}$ appearing in Lemma 2.4 are denoted A^{out} and are called the asymptotic fields. For $A = A^*$ these operators are essentially self-adjoint on $D(P^0)$ and their self-adjoint extensions are denoted by the same symbol. For any $\mathcal{O} \in \mathcal{K}$ we introduce the von Neumann algebra:

$$\mathcal{A}^{\mathrm{out}}(\mathcal{O}) := \{\, e^{iA^{\mathrm{out}}} \,:\, A \in \mathcal{A}^{C_*}(\mathcal{O}), \ A^* = A \,\}''. \tag{2.16}$$

The triple $(\mathcal{A}^{\mathrm{out}}, U, \Omega)$ satisfies all the properties from Definition 2.1, except, perhaps, for the cyclicity of the vacuum. If the latter property also holds, then we say that the theory $(\mathcal{A}, U, \Omega)$ is **asymptotically complete**. Clearly, for the definition of asymptotic completeness the case $f \equiv 1$ suffices. However, the operators $A^{\mathrm{out}}\{f\}$ for other choices of f will be needed in Sect. 5 at the technical level. For this reason we collected their properties above.

Now we are ready to state the main result of this paper:

Theorem 2.5. *Let $(\mathcal{A}, U, \Omega)$ be a local net containing massless particles with helicity zero or with helicities $(h, -h)$ for some $h \in \mathbb{N}$. If this net is asymptotically complete, then it satisfies the Bisognano-Wichmann property.*

Proof. Follows from Theorem 4.13 and Propositions 5.1, 5.4 below. $\qquad\square$

Even if the original net $(\mathcal{A}, U, \Omega)$ is not asymptotically complete, we can set $\mathfrak{H}^{\mathrm{out}} := \overline{\mathcal{A}^{\mathrm{out}}\Omega}$ and define the asymptotic net $(\mathcal{A}^{\mathrm{out}}|_{\mathfrak{H}^{\mathrm{out}}}, U|_{\mathfrak{H}^{\mathrm{out}}}, \Omega)$ which is asymptotically complete by construction. In view of the commutation relations (2.15), this net can be considered free, but it is not automatically the net of the corresponding textbook free field theory[1]. By Theorem 2.5, this net satisfies the Bisognano-Wichmann property if it contains massless Wigner particles with helicity zero or $(h, -h)$, $h \in \mathbb{N}$.

We note that the local nets satisfying the assumptions of Theorem 2.5 are a posteriori in the setting of [GL95]. Indeed, the modular covariance is an obvious consequence of the Bisognano-Wichmann property and the Reeh-Schlieder property for spacelike cones follows from the spectrum condition and cyclicity of the vacuum under $\mathcal{A}$ (cf. [Bu75, Appendix]). From the spin-statistics theorem of [GL95] it follows that such nets are actually covariant under $\mathcal{P}_+^{\uparrow}$. Furthermore, by the CPT theorems of this reference, the unitary representation U of $\mathcal{P}_+^{\uparrow}$ extends to an (anti-)unitary covariant representation of $\mathcal{P}_+$ (the group generated by $\mathcal{P}_+^{\uparrow}$ and the PT operator Θ) as follows:

Corollary 2.6. *Let $(\mathcal{A}, U, \Omega)$ be a local net as in Theorem 2.5. Then U extends to an (anti-)unitary representation of the Poincaré group $\mathcal{P}_+$ by*

$$J_{W_1} U(R_1(\pi)) = U(\Theta)$$

where $\Theta x = -x$ with $x \in \mathbb{R}^{1+3}$, J_{W_1} is the modular conjugation associated to $(\mathcal{A}(W_1), \Omega)$ and $R_1(\pi)$ is the rotation by π around the first axis.

3 Preliminaries

3.1 Scattering states of massless particles

In this subsection we provide some preparatory information about the Hilbert space of scattering states $\mathfrak{H}^{\mathrm{out}} := \overline{\mathcal{A}^{\mathrm{out}}\Omega}$ introduced above. Namely, we extract the creation and annihilation parts of the asymptotic fields (2.13) in order to facilitate the construction of scattering states. We still follow [AD17] which in turn relied here on [DH15]. Let

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[1] For example, if we choose as the original net the 'truncated' net, s.t. the local algebras of regions below certain size are declared to be $\mathbb{C}1$, the asymptotic net will inherit this property [AD17].

$\theta \in C^\infty(\mathbb{R})$, $0 \leq \theta \leq 1$, be supported in $(0, \infty)$ and equal to one on $(1, \infty)$. Moreover, let $\beta \in C_0^\infty(\mathbb{R}^4)$, $0 \leq \beta \leq 1$, be equal to one in some neighbourhood of zero and satisfy $\beta(-p) = \beta(p)$. Furthermore, for a parameter $1 \leq r < \infty$ and a future oriented timelike unit vector n we define

$$\widetilde{\eta}_{\pm,r}(p) := \theta(\pm r(n_\mu p^\mu))\beta(r^{-1}p), \tag{3.1}$$

where tilde denotes the Fourier transform. As $r \to \infty$ these functions approximate the characteristic functions of the positive/negative energy half planes $\{ p \in \mathbb{R}^4 : \pm n_\mu p^\mu \geq 0 \}$. We also have $\bar{\eta}_{\pm,r} = \eta_{\mp,r}$. Note that the family of functions $\eta_{\pm,r}$, as specified above, is invariant under Lorentz transformations.

Proposition 3.1. [Bu77, AD17] *Let $A \in \mathcal{A}_{C_*}$, $f \in C^\infty(S^2)$. Suppose that the timelike unit vectors n entering the definition of A and of $\eta_{\pm,r}$ coincide. Then:*

(a) *The limits $A^{\text{out}}\{f\}^\pm\Psi := \lim_{r\to\infty} A^{\text{out}}\{f\}(\eta_{\pm,r})\Psi$, $\Psi \in D_{P^0}$, exist and define the creation and annihilation parts of $A^{\text{out}}\{f\}$ as operators on D_{P^0}. $A^{\text{out}}\{f\}^\pm$ do not depend on the choice of the functions θ and β in (3.1) within the specified restrictions.*

(b) *$(A^{\text{out}}\{f\}^\pm)^*|_{D_{P^0}} = A^{*\text{out}}\{\bar{f}\}^\mp$. In particular, $A^{\text{out}}\{f\}^\pm$ are closable operators.*

(c) *$A^{\text{out}}\{f\}^\pm D_{P^0} \subset D_{P^0}$.*

(d) *$A^{\text{out}}\{f\} = A^{\text{out}}\{f\}^+ + A^{\text{out}}\{f\}^-$ on D_{P^0}.*

Making use of Proposition 3.1 and of (2.15) we also obtain on D_{P^0}

$$[A^{\text{out}}\{f\}^-, A'^{\text{out}}\{f'\}^+] = \langle A^{*\text{out}}\{\bar{f}\}^+\Omega, A'^{\text{out}}\{f'\}^+\Omega\rangle 1_{\mathfrak{H}} \tag{3.2}$$

and the commutators of pairs of creation (resp. annihilation) operators vanish. The following definition of scattering states is slightly more general than in [Bu77, AD17], as we do not assume $f \equiv 1$. The proof is an obvious application of the canonical commutation relations (3.2).

Proposition 3.2. [Bu77, AD17] *The states $\Psi^{\text{out}} := A_1^{\text{out}}\{f_1\}^+ \ldots A_n^{\text{out}}\{f_n\}^+\Omega$ have the following properties:*

(a) *Ψ^{out} depends only on the single-particle states $\Phi_i = A_i^{\text{out}}\{f\}\Omega \in \mathfrak{H}^{(1)}$. Therefore, we write $\Psi^{\text{out}} = \Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_n$.*

(b) *$\langle \Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_n, \Phi'_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi'_{n'}\rangle = \delta_{n,n'} \sum_{\sigma \in \mathfrak{S}_n} \langle \Phi_1, \Phi'_{\sigma_1}\rangle \ldots \langle \Phi_n, \Phi'_{\sigma_n}\rangle$, where $\mathfrak{S}_n$ is the set of all permutations of $(1, \ldots, n)$.*

The subspace of $\mathfrak{H}$ spanned by vectors of the form $\Psi^{\text{out}} = \Phi_1 \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} \Phi_n$ for fixed n will be denoted $\mathfrak{H}^{(n)}$. We note that

$$\mathfrak{H}^{\text{out}} := \overline{\mathcal{A}^{\text{out}}\Omega} = \bigoplus_{n\geq 0} \mathfrak{H}^{(n)}, \tag{3.3}$$

where $\mathfrak{H}^{(0)} = \mathbb{C}\Omega$ and $\mathfrak{H}^{(1)}$ was introduced in Definition 2.3. The last equality in (3.3) follows from density of vectors of the form (2.14) in $\mathfrak{H}^{(1)}$ and from the canonical commutation relations (3.2) by standard Fock space arguments. Clearly, $\mathfrak{H}^{\text{out}}$ is naturally isomorphic to the symmetric Fock space over $\mathfrak{H}^{(1)}$, denoted $\Gamma(\mathfrak{H}^{(1)})$.

3.2 Standard subspaces

We recall here some elements of the theory of standard subspaces following [Lo]. In the later part of this subsection we also provide several results which we were not able to find in literature and that will be needed in our investigation.

A real linear, closed subspace H of a complex Hilbert space $\mathcal{H}$ is called **cyclic** if $H + iH$ is dense in $\mathcal{H}$, **separating** if $H \cap iH = \{0\}$ and **standard** if it is cyclic and separating.

Given a standard subspace H the associated **Tomita operator** S_H is defined to be the closed anti-linear involution with domain $H + iH$, given by:

$$S_H : H + iH \ni \xi + i\eta \mapsto \xi - i\eta \in H + iH, \qquad \xi, \eta \in H.$$

The polar decomposition

$$S_H = J_H \Delta_H^{1/2}$$

defines the positive self-adjoint **modular operator** Δ_H and the anti-unitary **modular conjugation** J_H. Δ_H is invertible and $J_H \Delta_H J_H = \Delta_H^{-1}$.

Let H be a real linear subspace of $\mathcal{H}$, the *symplectic complement* of H is defined by

$$H' := \{\xi \in \mathcal{H} \ : \ \mathrm{Im}\langle \xi, \eta \rangle = 0, \forall \eta \in H\} = (iH)^{\perp_\mathbb{R}} \ ,$$

where $\perp_\mathbb{R}$ denotes the orthogonal complement in $\mathcal{H}$ with respect to the real part of the scalar product on $\mathcal{H}$. H' is a closed, real linear subspace of $\mathcal{H}$. It is a fact that H is cyclic (resp. separating) iff H' is separating (resp. cyclic), thus H is standard iff H' is standard and in this case

$$S_{H'} = S_H^* \ ,$$

with $J_{H'} = J_H$ and $\Delta_{H'} = \Delta_H^{-1} = J_H \Delta_H J_H$ [Lo]. Furthermore, if H is standard, then $H = H''$. We recall that the one-parameter, strongly continuous group $t \mapsto \Delta_H^{it}$ is called the **modular group** of H and

$$\Delta_H^{it} H = H, \quad J_H H = H' \ , \qquad t \in \mathbb{R} \ .$$

There is a 1-1 correspondence between Tomita operators and standard subspaces.

Proposition 3.3. [Lo]. *The map*

$$H \longmapsto S_H \tag{3.4}$$

is a bijection between the set of standard subspaces of $\mathcal{H}$ and the set of closed, densely defined, anti-linear involutions on $\mathcal{H}$.

The following are three basic results on standard subspaces.

Lemma 3.4. [Mo18]. *Let $H, K \subset \mathcal{H}$ be standard subspaces and $U \in \mathcal{U}(\mathcal{H})$ be a unitary operator on $\mathcal{H}$ such that $UH = K$. Then $U\Delta_H U^* = \Delta_K$ and $UJ_H U^* = J_K$.*

Lemma 3.5. [Lo]. *Let $H \subset \mathcal{H}$ be a standard subspace, and $K \subset H$ be a closed, real linear subspace of H. If $\Delta_H^{it} K = K$, $\forall t \in \mathbb{R}$, then K is a standard subspace of $\mathcal{K} := \overline{K + iK}$ and $\Delta_H|_K$ is the modular operator of K on $\mathcal{K}$. Moreover, if K is a cyclic subspace of $\mathcal{H}$, then $H = K$.*

Theorem 3.6. [Lo]. *Let $H \subset \mathcal{H}$ be a standard subspace, and $U(t) = e^{itP}$ be a one-parameter unitary group on $\mathcal{H}$ with a generator $\pm P > 0$, such that $U(t)H \subset H$, $\forall t \geq 0$. Then,*

$$\begin{cases} \Delta_H^{is} U(t) \Delta_H^{-is} = U(e^{\mp 2\pi s} t) \\ J_H U(t) J_H = U(-t) \end{cases} \qquad \forall t, s \in \mathbb{R}. \qquad (3.5)$$

We note that the above result is a variant of the Borchers theorem [Bo92, Fl98] for standard subspaces.

The following three lemmas, which we could not find in the literature, will be needed to analyze the subspaces $H^{(1)}(W)$ defined in (5.2) below. They concern decompositions of standard subspaces w.r.t. projections E commuting with S_H. Since S_H is unbounded and not self-adjoint, we mean here that E commutes with J_H and bounded Borel functions of Δ_H. If $\xi' \in D(S_H^*) = H' + iH'$ and $\xi \in D(S_H) = H + iH$, then, for such E

$$\langle \xi', S_H E \xi \rangle = \overline{\langle S_H^* \xi', E\xi \rangle} = \overline{\langle J_H \Delta_H^{-1/2} \xi', E\xi \rangle} = \lim_{n \to \infty} \langle \chi_n(\Delta_H^{-1/2}) \Delta_H^{-1/2} \xi', E J_H \xi \rangle$$

$$= \lim_{n \to \infty} \langle \xi', E\chi_n(\Delta_H^{-1/2}) \Delta_H^{-1/2} J_H \xi \rangle = \langle \xi', E\Delta_H^{-1/2} J_H \xi \rangle = \langle \xi', E S_H \xi \rangle, \quad (3.6)$$

where χ_n is the characteristic function of $[-n, n]$ and we made use of the fact that $\xi', J_H \xi \in D(\Delta_H^{-1/2})$ to control the limit $n \to \infty$.

Lemma 3.7. *Let $H \subset \mathcal{H}$ be a standard subspace and $E = E^2 = E^*$ be a projection commuting with S_H. Then $H = EH \oplus (1-E)H$. Furthermore, EH and $(1-E)H$ are standard in $E\mathcal{H}$ and $(1-E)\mathcal{H}$, respectively.*

Proof. H is defined to be the kernel of $1 - S_H$. Now since E commutes with S_H, for every $\xi \in H$, $E\xi \in \text{Ker}(1 - S_H)$, thus $E\xi \in H$ (cf. computation (3.6) above). As the same argument applies to $(1 - E)$ and $\xi = E\xi + (1-E)\xi$, we have the claim. The last statement is obvious. $\square$

Lemma 3.8. *Let $H \subset \mathcal{H}$ be a standard subspace and $E = E^2 = E^*$ be a projection commuting with S_H. Then $H' = EH' \oplus (1-E)H'$. Furthermore, EH' and $(1-E)H'$ are standard in $E\mathcal{H}$ and $(1-E)\mathcal{H}$, respectively.*

Proof. If $E \in \mathcal{B}(\mathcal{H})$ is a projection commuting with S_H, then E also commutes with $S_{H'} = J_H \Delta_H^{-1/2} = \Delta_H^{1/2} J_H$ and the decomposition $H' = EH' \oplus (1-E)H'$ follows as in Lemma 3.7. We note that S_H and $S_{H'}$ and their polar decompositions decompose through E. (Clearly, $S_{EH} = S_H|_{E\mathcal{H}}$ and $S_{(1-E)H} = S_H|_{(1-E)\mathcal{H}}$). Consequently, $(EH)' = EH'$ and $((1-E)H)' = (1-E)H'$ on $E\mathcal{H}$ and $(1-E)\mathcal{H}$, respectively. Since EH and $(1-E)H$ are cyclic and separating in $E\mathcal{H}$ and $(1-E)\mathcal{H}$, the claim follows. $\square$

An immediate consequence is:

Lemma 3.9. *Let $H, K \subset \mathcal{H}$ standard subspaces and E a projection satisfying the assumptions of Lemma 3.8 w.r.t. H and K. Assume that $K \subset H'$. Then $EK \subset EH'$ and $(1-E)K \subset (1-E)H'$.*

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Proof. Since $H' = EH' \oplus (1-E)H'$ and $K = EK \oplus (1-E)K$, for every $\xi \in K$ we have $E\xi \in K$, thus $E\xi \in EH'$. We conclude that $EK \subset EH'$ and analogously $(1-E)K \subset (1-E)H'$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3.3   One particle nets

Let U be a unitary representation of the Poincaré group $\mathcal{P}_+^\uparrow$ on a Hilbert space $\mathcal{H}$. We shall call a U**-covariant (or Poincaré covariant) net of standard subspaces on wedges** a map

$$H : \mathcal{W} \ni W \longmapsto H(W) \subset \mathcal{H},$$

associating to every wedge in $\mathbb{R}^{1+3}$ a closed real linear subspace of $\mathcal{H}$, satisfying the following properties[2]:

1. **Isotony:** If $W_1, W_2 \in \mathcal{W}$ and $W_1 \subset W_2$ then $H(W_1) \subset H(W_2)$;

2. **Poincaré covariance:** $U(g)H(W) = H(gW)$, $\forall g \in \mathcal{P}_+^\uparrow$, $\forall W \in \mathcal{W}$;

3. **Positivity of the energy:** the joint spectrum of translations in U is contained in the forward lightcone $V_+ = \{\, p \in \mathbb{R}^{1+3} : p^0 \geq 0, p^2 = (p,p) \geq 0 \,\}$;

4. **Cyclicity:** if $W \in \mathcal{W}$, then $H(W)$ is a cyclic subspace of $\mathcal{H}$;

5. **Locality:** if $W_1 \subset W_2'$ then $H(W_1) \subset H(W_2)'$.

We shall indicate a U-covariant net H of standard subspaces on wedges satisfying 1.-5. with the couple (U, H). This is the setting in which we are going to study the following property:

6. **Bisognano-Wichmann property**: if $W \in \mathcal{W}$, then $U(\lambda_W(2\pi t)) = \Delta_{H(W)}^{-it}$, $\forall t \in \mathbb{R}$;

The next property is a completeness property for a model in the sense of the causal structure and, by Lemma 3.5, is a consequence of the locality and the B-W properties (see e.g. [Mo18]).

7.   **Duality property**: if $W \in \mathcal{W}$, then $H(W)' = H(W')$.

Denote by $P^0, \boldsymbol{P}$ be the generators of translations in the representation U and $M = \sqrt{(P^0)^2 - \boldsymbol{P}^2}$ the resulting mass operator. Then Theorem 3.6 has the following corollary, which is well known in the context of nets of von Neumann algebras.

Corollary 3.10. *For any wedge W, the mass operator M commutes strongly[3] with $\Delta_{H(W)}$ and $J_{H(W)}$. Its real bounded Borel functions commute weakly with $S_{H(W)}$ on domains specified as in (3.6).*

---

[2] The notation W_1, W_2 in this definition should not be confused with the standard wedges in the direction of particular axes, as used in Sect. 2.1.

[3] Taking anti-linearity of $J_{H(W)}$ into account, commutation with *real* bounded Borel functions of M is understood here.

Proof. Consider the wedge W_1, defined as in Sect. 2.1, and the associated standard subspace $H(W_1)$. Translations in direction of the axes x_2 and x_3 fix W_1. In particular the generators of the associated translation group P_2 and P_3, respectively, commute strongly with $\Delta_{H(W_1)}$ and $J_{H(W_1)}$ by Lemma 3.4. Lightlike translations of the form $a_\pm(t) = (\pm t, t, 0, 0)$ with $t \geq 0$ have generators $P_\pm := (\pm P_0 - P_1)$ s.t. $\pm P_\pm \geq 0$ and $U(a_\pm(t))H(W_1) \subset H(W_1)$ for $t \geq 0$. By the Borchers theorem for standard subspaces (Theorem 3.6) we have that $U(a_\pm)$ have the commutation relations as in equation (3.5):

$$\Delta^{is}_{H(W_1)} U(a_\pm(t)) \Delta^{-is}_{H(W_1)} = U(a_\pm(e^{\mp 2\pi s}t)) \Rightarrow \Delta^{is}_{H(W_1)} f(P_\pm) \Delta^{-is}_{H(W_1)} = f(e^{\mp 2\pi s} P_\pm)$$

$$J_{H(W_1)} U(a_\pm(t)) J_{H(W_1)} = U(a_\pm(-t)) \Rightarrow J_{H(W_1)} f(P_\pm)^* J_{H(W_1)} = f(P_\pm),$$

where f is any bounded Borel function. The implications above follow by approximating f pointwise with Schwartz-class functions (which gives strong convergence of the corresponding operators) and using the Fourier transform. Now $P^2 = M^2 = -(P_+ P_- + P_2^2 + P_3^2)$ and for any real Borel function g it is easy to check, using the above relations, that $g(M^2)$ commutes with $\Delta_{H(W)}$ and $J_{H(W)}$. Indeed, by approximating g pointwise by Schwartz-class functions, applying the Fourier transform and using that P_2, P_3 commute strongly with $\Delta_{H(W)}$ and $J_{H(W)}$ it suffices to verify that

$$\Delta^{is}_{H(W_1)} e^{-iP_+ P_- t} \Delta^{-is}_{H(W_1)} = e^{-iP_+ P_- t}, \quad J_{H(W_1)} e^{-iP_+ P_- t} J_{H(W_1)} = e^{iP_+ P_- t}.$$

This is achieved by approximating $e^{-iP_+ P_- t}$ pointwise by linear combinations of expressions of the form $f_+(P_+)f_-(P_-)$, where f_+, f_- are bounded Borel functions, and applying the relations above.

For a general wedge W, let $g \in \mathcal{P}^\uparrow_+$ s.t. $W = gW_1$. Then, by Lemma 3.4, $J_{H(W)} = U(g)J_{H(W_1)}U(g)^*$, $\Delta_{H(W)} = U(g)\Delta_{H(W_1)}U(g)^*$ and thus $S_W = U(g)S_{H(W_1)}U(g)^*$. Clearly $P^2 = U(g)P^2 U(g)^*$, thus P^2 commutes with $J_{H(W)}, \Delta_{H(W)}, S_{H(W)}$ for every $W \in \mathcal{W}$ in the same sense as discussed above. $\qquad\square$

3.4 Induced representations: the Poincaré group and its subgroups

Our group theoretic considerations in the remaining part of Sect. 3 and in Sect. 4 are based on the **standing assumption** that all the representations of topological groups on Hilbert spaces are strongly continuous.

Let G be a locally compact group, N a nontrivial closed normal abelian subgroup and H another closed subgroup such that $G = N \rtimes H$[4]. Assume that the action of G on $\hat{N}$, the dual group of N, obtained by conjugation, is regular (cf. [Fol16] Sect. 6.6 and Definition C.1). Let $p \in \hat{N}$, Ω_p be the orbit under the G-dual action[5], with

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[4] We warn the reader, that the letter H, used earlier for nets of standard subspaces, is now used for groups. As we will not consider nets of standard subspaces in the remaining part of Sect. 3, there is no risk of confusion.

[5] $\chi_{g^{-1}p}(x) = \chi_p(gxg^{-1})$.

$x \in N$, $p \in \hat{N}$ and $g \in G$, Stab_p and $\overline{\mathrm{Stab}}_p$ be the stabilizers of the point p under the action of H and G. Stab_p is called the *little group*. Let χ_p be the character associated to $p \in \hat{N}$.

Every unitary irreducible representation of G is obtained by induction in the following way (see e.g. [Fol16] Sect. 6)

$$\mathrm{Ind}^G_{\overline{\mathrm{Stab}}_p}(\chi_p \cdot V), \tag{3.7}$$

where V and $\chi_p \cdot V$ are unitary representations of the little group Stab_p and of $\overline{\mathrm{Stab}}_p$, respectively, and the following proposition holds:

Proposition 3.11. *[Fol16] Let $G = N \rtimes H$ as above. Every unitary irreducible representation of G is equivalent to one of the form* (3.7). *Furthermore* $\mathrm{Ind}^G_{\overline{\mathrm{Stab}}_p}(\chi_p \cdot V)$ *and* $\mathrm{Ind}^G_{\overline{\mathrm{Stab}}_q}(\chi_q \cdot V')$ *are equivalent if and only if p and q belongs to the same orbit, say $p = g\,q$, and V and $V' \circ \mathrm{ad}_{g^{-1}}$ are equivalent representations of Stab_p for some $g \in G$.*

If $W = \mathrm{Ind}^G_{\overline{\mathrm{Stab}}_p}(\chi_p \cdot V)$ is an irreducible representation of G then the spectral measure of $W|_N$ is concentrated on the orbit $o = Gp$ (cf. Proposition 6.36 [Fol16]). References for general induced representations are for instance [Fol16, Ki76, BR86].

The Poincaré group. The Minkowski space $\mathbb{R}^{1+3}$ is the 4-dimensional real vector space endowed with the metric tensor $\eta = \mathrm{diag}(1, -1, -1, -1)$. The Lorentz group $\mathcal{L}$ is the group of linear transformations L s.t. $L^T \eta L = \eta$. Let $\mathcal{L}^\uparrow_+$ be the connected component of the identity of the Lorentz group and $\widetilde{\mathcal{L}}^\uparrow_+ = \mathrm{SL}(2, \mathbb{C})$ its universal covering group. Let $\widetilde{\mathcal{P}}^\uparrow_+ = \mathbb{R}^{1+3} \rtimes \mathrm{SL}(2, \mathbb{C})$ be the universal covering of the Poincaré group $\mathcal{P}^\uparrow_+ = \mathbb{R}^{1+3} \rtimes \mathcal{L}^\uparrow_+$ (the inhomogeneous symmetry group of $\mathbb{R}^{1+3}$) and Λ be the covering map. First of all, we recall that to any 4-vector is 1-1 associated a 2×2 matrix

$$x_\sim = x_0\,\mathbf{1} + \sum_{i=1,2,3} x_i\,\sigma_i = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix},$$

where σ_i are the Pauli matrices. Real vectors define Hermitian matrices. If $A \in \mathrm{SL}(2, \mathbb{C})$, and $\Lambda : \widetilde{\mathcal{P}}^\uparrow_+ \to \mathcal{P}^\uparrow_+$ is the covering homomorphism, then the Poincaré action is ruled by the following relation

$$(\Lambda(A)x)_\sim = Ax_\sim A^*. \tag{3.8}$$

Let U be a unitary strongly continuous representation of the Poincaré group, then the representation of an x-translation has the form $U(x) = e^{iPx}$ where P is a vector of four self-adjoint operators and Px is obtained through the Minkowski product. Every $g \in \widetilde{\mathcal{L}}^\uparrow_+$ acts on an x-translation by the adjoint action, namely $gxg^{-1} = \Lambda(g)x$. Let $\mathrm{Sp}(P)$ be the joint spectrum of generators of translations and p be a point in the spectrum, then we have the character $\chi_p(x) = e^{ipx}$. As in the general case, the dual action on the momentum space is defined s.t. $\chi_p(\Lambda(g) \cdot x) = \chi_{p'}(x)$ and it is easy to see that $p' = \Lambda(g)^{-1}p$, where the latter is the matrix-vector multiplication. Clearly

the adjoint action of the translations act trivially on themselves, hence on their dual (see e.g. [BR86]).

Positive energy massless representations of the Poincaré group. Let χ_q, $q \neq 0$, be a character of the translation group. We shall call Stab_q and $\overline{\mathrm{Stab}}_q$ the stabilizers of the point q through the $\widetilde{\mathcal{L}}_+^\uparrow$ and $\widetilde{\mathcal{P}}_+^\uparrow$ actions, respectively. The latter is $\overline{\mathrm{Stab}}_q = \mathbb{R}^{1+3} \rtimes \mathrm{Stab}_q$, where Stab_q shall be called as above the little group. Any massless, unitary, positive energy representation of $\widetilde{\mathcal{P}}_+^\uparrow$ is obtained starting with the character associated to $q := (1, 1, 0, 0) \in \partial V_+$ ($\partial V_+ \setminus \{0\}$ is an $\mathcal{L}_+^\uparrow$-orbit) and inducing by a unitary representation of the $\overline{\mathrm{Stab}}_q$ group. Note that a $\overline{\mathrm{Stab}}_q$ representation is of the form

$$\mathbb{R}^{1+3} \rtimes \mathrm{Stab}_q \ni (x, \sigma) \mapsto \chi_q(x) V(\sigma),$$

where V is the unitary representation of Stab_q. The little group Stab_q is isomorphic to $\widetilde{\mathrm{E}}(2) = \mathbb{R}^2 \rtimes \mathbb{T}$, where $\mathbb{T}$ is the unit circle. Note that $r_1(\theta)$ generate $\mathbb{T}$. $\widetilde{\mathrm{E}}(2)$ is the double cover of $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, which is the group of Euclidean motions in two dimensions[6]. Irreducible representations V of $\widetilde{\mathrm{E}}(2)$ fit in one of the following two classes: (See e.g. [Va85] and [BR86, page 520])

(a) The restriction of V to $\mathbb{R}^2$ is trivial;

(b) The restriction of V to $\mathbb{R}^2$ is non-trivial.

Irreducible representations of $\widetilde{\mathrm{E}}(2)$ in class (a) are labelled by half-integers h, called the *helicity* parameters. Irreducible representations in class (b) are labelled by $\kappa > 0$, the radius of a circle in $\mathbb{R}^2$, namely the joint spectrum of the $\widetilde{\mathrm{E}}(2)$-translations, and a Bose/Fermi alternative parameter $\epsilon \in \{0, \frac{1}{2}\}$.

Let

$$U = \mathrm{Ind}_{\overline{\mathrm{Stab}}_q}^{\widetilde{\mathcal{P}}_+^\uparrow} (\chi_q \cdot V)$$

be a unitary representation of $\widetilde{\mathcal{P}}_+^\uparrow$ induced from the representation $\chi_q \cdot V$ of $\overline{\mathrm{Stab}}_q$. We say that U has *finite helicity* or *infinite spin* if V has the form (a) or (b), respectively.

An irreducible **finite helicity** representation is of the form

$$U_h = \mathrm{Ind}_{\overline{\mathrm{Stab}}_q}^{\widetilde{\mathcal{P}}_+^\uparrow} (\chi_q \cdot V_{2h}), \qquad h \in \frac{\mathbb{Z}}{2}, \tag{3.9}$$

where $V_{2h}(y, g) = (2h)(g)$, $(y, g) \in \mathbb{R}^2 \rtimes \mathbb{T}$ and $2h$ is the one dimensional representation of $\mathbb{T}$ of character $h \in \mathbb{Z}/2$. In particular, $r_1 \in \mathrm{Stab}_q$ and $(2h)(r_1(\varphi)) = e^{i\frac{\varphi}{2} 2h}$ for a $\varphi \in [0, 4\pi]$. Integer and half-integer values of h discern, respectively, bosonic and fermionic representation of $\widetilde{\mathrm{E}}(2)$, hence, by induction, of $\widetilde{\mathcal{P}}_+^\uparrow$.

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[6]We note that $\widetilde{\mathrm{E}}(2)$ is not the universal covering group of $\mathrm{E}(2)$, as it is not simply connected. In particular the covering map is given by $\mathbb{T} \ni r_1(\theta) = e^{i\frac{\theta}{2}\sigma_1} \mapsto e^{i\theta} \in \mathrm{SO}(2)$, where we have identified $\mathbb{R}^2$ with $\mathbb{C}$ and rotations as multiplication by a phase $e^{i\phi}$.

3.5 G_W and related subgroups of the Poincaré group and their representations

In this subsection we introduce certain subgroups of $\widetilde{\mathcal{P}}_+^\uparrow$ which will be needed in Sect. 4 below to formulate a criterion for the B-W property. We refer to Sect. 2.1 for the definitions of the wedges W_i, $i = 1, 2, 3$, and the set of wedges $\mathcal{W}$. We also recall from this subsection that $R_i, \Lambda_i \in \mathcal{L}_+^\uparrow$ and $r_i, \lambda_i \in \mathrm{SL}(2, \mathbb{C})$ denote the one-parameter families of rotations and boosts preserving the wedges W_i, $i = 1, 2, 3$.

Definition 3.12. *We denote with*

- *G_3^0 the subgroup of $A \in \mathrm{SL}(2, \mathbb{C})$ s.t. $\Lambda(A)W_3 = W_3$.*

- *$G_3 = \langle G_3^0, \mathbb{R}^{1+3} \rangle$, where $\mathbb{R}^{1+3}$ is the translation group and $\langle G_3^0, \mathbb{R}^{1+3} \rangle$ denotes the group generated by G_3^0 and $\mathbb{R}^{1+3}$.*

- *$\tilde{G}_3^0 = \langle G_3^0, r_1(\pi) \rangle$, $\tilde{G}_3 = \langle G_3, r_1(\pi) \rangle = \mathbb{R}^{1+3} \rtimes \tilde{G}_3^0$.*

For a general wedge $W \in \mathcal{W}$, G_W^0, G_W, $\tilde{G}_W$ and $\tilde{G}_W^0$ are defined by the transitive action of $\mathcal{P}_+^\uparrow$ on wedges. We will denote the massless orbits of the $\mathbb{R}^{1+3}$ translation characters under the G_3 action with $\sigma_r = \{p = (p_0, \boldsymbol{p} = (p_1, p_2, p_3)) \in \mathbb{R}^{1+3} : p_1^2 + p_2^2 = r^2, p_0^2 = \boldsymbol{p}^2, p_0 > 0\}$, $\sigma_0^\pm = \{p \in \mathbb{R}^{1+3} : p_1 = 0 = p_2, p_0 = \pm p_3, p_0 > 0\}$ and $\sigma_0 = \sigma_0^+ \cup \sigma_0^-$. In the present paper, we will be interested in orbits σ_r with $r > 0$ since $\sigma_0^\pm$ have null measure w.r.t. the Lorentz invariant measure on ∂V_+. Finally, we warn the reader that $\tilde{G}_3, \tilde{G}_W$ do not denote the covering groups of G_3, G_W.

Note that $G_3^0 = \langle \lambda_3, r_3, r(2\pi) \rangle$. Indeed, any $\mathrm{SL}(2, \mathbb{C})$ element implementing a Poincaré transformation can be decomposed by the polar decomposition $A = U_A \cdot T_A$ (see e.g. [Mor06]), where U_A is a rotation and T_A a boost. Let $\Lambda : \mathrm{SL}(2, \mathbb{C}) \to \mathcal{L}_+^\uparrow$ be the covering map, then $\Lambda(A)W_3 = W_3$ iff $U_A^{-1}W_3 = T_AW_3$. Assume that there exists a transformation A such that $U_A^{-1}W_3 \neq W_3 \neq T_AW_3$ but $U_A^{-1}W_3 = T_AW_3$. Consider the edge of the wedge $E := \{x \in \mathbb{R}^{1+3} : x_0 = 0 = x_3\}$. Then, $U_A^{-1}E = \{x \in \mathbb{R}^{1+3} : x_0 = 0 = (U_A x)_3\}$ cannot be equal to $T_A E = \{x \in \mathbb{R}^{1+3} : (T_A^{-1}x)_0 = 0 = (T_A^{-1}x)_3\}$. In particular $U_A^{-1}W_3 = W_3 = T_AW_3$.

Next, we note that G_3^0 and $\tilde{G}_3^0$ share the same orbits in ∂V_+ as the following remark explains.

Remark 3.13. *[Mo18] Fix $p = (p_0, p_1, p_2, p_3) \in (\partial V_+ \smallsetminus \{0\}) := \{p \in \mathbb{R}^{1+3} : p^2 = 0, p_0 > 0\}$. The $R_1(\pi)$-rotation*

$$R_1(\pi)p = (p_0, p_1, -p_2, -p_3)$$

can be obtained as a composition of a Λ_3-boost of parameter t_p and a R_3-rotation of parameter θ_p as

$$\Lambda_3(t_p)R_3(\theta_p)(p_0, p_1, p_2, p_3) = \Lambda_3(t_p)(p_0, p_1, -p_2, p_3) = (p_0, p_1, -p_2, -p_3) \qquad (3.10)$$

for all the orbits except for $\sigma_0^\pm$, where $\sigma_0^\pm$ appeared in Definition 3.12. Clearly t_p and θ_p depend on p and the orbits excluded by this geometrical fact have null measure w.r.t. the Lorentz invariant measure on ∂V_+. The discussion does not change if p is considered as an element of $\mathbb{R}^{1+3}$ or of its dual.

By (3.10), we deduce that almost all G_3^0 orbits on ∂V_+ are preserved by the $R_1(\pi)$-action. Furthermore, we note that $R_1(\pi)$ sends W_3 onto W_3'. Thus any transformation $R \in \mathcal{P}_+^\uparrow$ such that $RW_3 = W_3'$ also preserves the G_3^0 orbits on ∂V_+ as well as $R_1(\pi)$, since $R_1(\pi)R \in G_3^0$. We have just seen that it is possible to pointwise reconstruct a transformation sending W to W' just starting with elements in G_W^0. With the help of the modularity condition (cf. Definition 4.1 and Theorem 4.2 below), this gives the proof of the B-W property in the scalar case in [Mo18].

As the regularity of the action of G_3^0 and $\tilde{G}_3^0$ on $\mathbb{R}^{1+3}$ is verified in Appendix C, we can apply the theory of induced representations to $G_3 = \mathbb{R}^{1+3} \rtimes G_3^0$ and $\tilde{G}_3 = \mathbb{R}^{1+3} \rtimes \tilde{G}_3^0$. Choose a point q_r on each massless, positive energy orbit σ_r of G_3^0 on the dual of $\mathbb{R}^{1+3}$. Up to a null measure set in ∂V_+, the stabilizer of q_r in G_3 is $\mathbb{R}^{1+3} \times \langle r(2\pi) \rangle$ (cf. the definition of the orbits σ_r and of G_3^0). Thus there exist only two irreducible representations of G_3 induced by χ_{q_r}, namely

$$W_{r,n} = \operatorname{Ind}_{\mathbb{R}^{3+1} \times \langle r(2\pi) \rangle}^{G_3}(\chi_{q_r} \cdot V_n), \tag{3.11}$$

where $V_n(r(2\pi)) = (-1)^n$ and $n = 0, 1$, cf. Proposition 3.11. They correspond to bosonic and fermionic representations of G_3.

Again by Remark 3.13 the subgroups $\tilde{G}_3$ and $\tilde{G}_3^0$ share the same orbits σ_r with $r > 0$ on ∂V_+ (up to a null measure set in ∂V_+). On the other hand the stabilizer of the point $q_r = (r, r, 0, 0) \in \sigma_r$ in $\tilde{G}_3$ is $\mathbb{R}^{1+3} \rtimes \langle r_1(\pi) \rangle$.

Thus the little group of q_r, the subgroup of $\tilde{G}_3^0$ fixing q_r, is $\mathbb{Z}_4$. We have four irreducible representations of $\mathbb{Z}_4$ indexed by the representation of the generator, namely $V_n(r_1(\pi)) = i^n$, with $n = 0, 1, 2, 3$. Correspondingly, we have four induced representations of $\tilde{G}_3$ associated to each orbit, namely

$$W_{r,n} = \operatorname{Ind}_{\mathbb{R}^{1+3} \rtimes \mathbb{Z}_4}^{\tilde{G}_3}(\chi_{q_r} \cdot V_n) \tag{3.12}$$

acting on the Hilbert spaces $\mathcal{H}_{r,n}$. Note that we called V_{2h} the representation of character $2h$, trivial on translations of $\widetilde{\mathrm{E}}(2)$, and V_n the representation of character n of $\mathbb{Z}_4$. That this is not an abuse of notation is justified by the fact that $r_1(\pi) \in \widetilde{\mathrm{E}}(2)$ and when we restrict V_{2h} to the group $\langle r_1(\pi) \rangle$, we get

$$V_{2h}(r_1(\pi)) = e^{i\frac{\pi}{2}2h} = i^{2h} = V_n(r_1(\pi)), \quad V_{2h}(r(2\pi)) = (-1)^n = V_n(r(2\pi))$$

and it is enough to consider $2h$ in $\mathbb{Z}_4$ in the first case or $2h$ in $\mathbb{Z}_2$ in the second. Furthermore $W_{r,n}$ as a representation of $\tilde{G}_3$ restricts to $W_{r,m}$ with $m \equiv n \pmod 2$ as a representation of G_3, cf. Lemma 4.9 in the next section. We will refer to (3.11) and (3.12) as *massless* representation of G_3 and $\tilde{G}_3$, respectively, since the translation spectrum lies on the boundary of the forward lightcone.

4 Modularity condition and U_h restriction

In this section we show that any local net of standard subspaces, covariant under a finite or infinite multiple of $U_h \oplus U_{-h}$, $h \in \mathbb{Z}$, satisfies the B-W property. The analysis is based on the following modularity condition:

Definition 4.1. *[Mo18] A (unitary, positive energy)* $\widetilde{\mathcal{P}}_+^\uparrow$ *representation is said to be* **modular** *if, for any U-covariant net of standard subspaces H, we have that (U, H) satisfies the B-W and duality properties.*

Let $W \in \mathcal{W}$. A unitary, positive energy $\widetilde{\mathcal{P}}_+^\uparrow$-representation U satisfies the **modularity condition** *if for an element $r_W \in \widetilde{\mathcal{P}}_+^\uparrow$ such that $\Lambda(r_W)W = W'$ we have that*

$$U(r_W) \in U(G_W)''. \tag{MC}$$

Note that (MC) depends neither on the choice of r_W nor on W. Indeed if $\tilde{r}_W \in \widetilde{\mathcal{P}}_+^\uparrow$ is another transformation such that $\Lambda(\tilde{r}_W)W = W'$ then $r_W \cdot \tilde{r}_W \in G_W$ and if (MC) holds for $U(r_W)$, then it holds for $U(\tilde{r}_W)$. By transitivity of the $\widetilde{\mathcal{P}}_+^\uparrow$ action on wedges it is not restrictive to fix a wedge region W. Condition (MC) can be straightforwardly stated when just a representation of $\tilde{G}_3$ is taken into account.

Theorem 4.2. *[Mo18] Let U be a positive energy unitary representation of the Poincaré group $\widetilde{\mathcal{P}}_+^\uparrow$. If condition (MC) holds for U, then any local U-covariant net of standard subspaces H, namely any pair (U, H), satisfies B-W and duality properties. In particular U is modular.*

We remark that the class of nets in Theorem 4.2, transforming under $\widetilde{\mathcal{P}}_+^\uparrow$, is more general than the class of bosonic nets we defined in Sect. 3.3. The theorem applies to the families of Poincaré representations covered by the following two results.

Proposition 4.3. *[Mo18] Let U be an irreducible scalar massive or massless representation of the Poincaré group, then U satisfies (MC).*

The proof of the above proposition adapts in irreducible finite helicity case. This section contains an alternative independent proof of modularity condition for finite helicity representations, see Corollary 4.11.

Proposition 4.4. *Let U, U_1 be unitary positive energy representations of $\tilde{G}_3$ on $\mathcal{H}$, satisfying (MC).*

(i) Let E be the projection on the subspace, where $U(r(2\pi)) = 1$. Then U satisfies (MC) iff both $EU(\cdot)E$ and $(1 - E)U(\cdot)(1 - E)$ satisfy (MC).

(ii) Let $\mathcal{K}$ be a Hilbert space, then (MC) holds for $U \otimes 1_{\mathcal{K}} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

(iii) Let U_2 be a unitary representation of $\tilde{G}_3$ s.t. U_2 is unitarily equivalent to U_1. Then U_2 satisfies (MC).

Proof. (i) Since $r(2\pi)$ commutes with every element in $\tilde{G}_3$, the spectral projection E of $U(r(2\pi))$ decomposes U into disjoint representations $U(\cdot) = EU(\cdot) \oplus (1 - E)U(\cdot)$. The thesis follows since $U(G_3)'' = (EU(G_3))'' \oplus ((1 - E)U(G_3))''$.

(ii) Is proved in [Mo18].

(iii) Let W be a unitary s.t. $WU_2(g)W^* = U_1(g)$, $g \in \tilde{G}_3$. It is easy to see that $WU_2(G_3)'W^* = U_1(G_3)'$ and $WU_2(G_3)''W^* = U_1(G_3)''$. If $WU_2(r_1(\pi))W^* = U_1(r_1(\pi))$, then

$$U_1(r_1(\pi)) \in U_1(G_3)'' \Rightarrow WU_2(r_1(\pi))W^* \in WU_2(G_3)''W^* \Rightarrow U_2(r_1(\pi)) \in U_2(G_3)'',$$

which concludes the proof. $\square$

The representations $W_{r,n}$ of $\tilde{G}_3$, restricted to G_3, give $W_{r,m}$, where $m \equiv n (\mathrm{mod}\ 2)$, see Lemma 4.9 below. Thus they are disjoint for different r, by the disjointness of the respective orbits σ_r, cf. Proposition 3.11. Furthermore, since $W_{r,n}$ is irreducible, $W_{r,n}(G_3)' = \mathbb{C} \cdot \mathbf{1}$ and $W_{r,n}(r_1(\pi)) \in W_{r,n}(G_3)'' = B(\mathcal{H}_{r,n})$. We deduce that $W_{r,n}$ satisfies (MC).

Corollary 4.5. *Let $W_{r,n}$ be the irreducible $\tilde{G}_3$-representation of radius $r > 0$. Then $W_{r,n}$ satisfies (MC).*

The following proposition ensures that also a direct integral of massless $\tilde{G}_3$ representations satisfies (MC).

Proposition 4.6. *Let μ be a positive Borel measure on $\mathbb{R}^+$. Assume that U_r are multiples of the massless $W_{r,n}$, $n = 0, 1, 2, 3$, representations of $\tilde{G}_W$. Then $U = \int_{\mathbb{R}^+}^{\oplus} U_r d\mu(r)$ satisfies (MC).*

One can reduce the argument to the cases $U_r|_{G_W} = W_{r,0}$ or $U_r|_{G_W} = W_{r,1}$ by Proposition 4.4 (i). We give details on the proof in Appendix B.

Now we want to verify the modularity condition (MC) for a large family of massless bosonic representations of the physically relevant form $U_h \oplus U_{-h}$ for integer helicities h. In order to prove the result, we want to disintegrate the restriction of U_h to the $\tilde{G}_3$ subgroup, to check the condition (MC) on the disintegration and to apply Theorem 4.2. The Poincaré representations are obtained by induction and the Mackey subgroup theorem teaches how to make the disintegration for such kind of representation.

Let H_1 and H_2 be subgroups of a locally compact group G. Then $H_1 \backslash G / H_2$ is the double coset, i.e., the set of the equivalence classes $[g] = H_1 g H_2$, with $g \in G$.

Definition 4.7. *[Ma52] Let G be a separable locally compact group.*

Closed subgroups H_1 and H_2 of G are said to be regularly related if there exists a sequence $E_0, E_1, E_2, \ldots$ of measurable subsets of G each of which is a union of double cosets in $H_1 \backslash G / H_2$ such that E_0 has Haar measure zero and each double coset not in E_0 is the intersection of the E_j which contain it.

Because of the correspondence between orbits of G/H_2 under H_1 and double cosets $H_1 \backslash G / H_2$, H_1 and H_2 are regularly related if and only if the orbits, (i.e., the double cosets) outside of a certain set of measure zero form the equivalence classes of a measurable equivalence relation. Given a topological standard measure space X, an equivalence relation $\sim$ and the quotient map $s : X \to Y = X/ \sim$, the equivalence relation is said to be measurable if there exists a countable family $\{F_n\}_{n \in \mathbb{N}}$ of subsets of the quotient space Y, s.t. $s^{-1}(F_n)$ is measurable and each point in Y is the intersection of all the $F_{n'}$, $n' \in \mathbb{N}$, containing this point.

Consider the map $s : G \to H_1 \backslash G / H_2$ carrying each element of G into its double coset. Then equip $H_1 \backslash G / H_2$ with the quotient topology given by s and consider a finite measure μ on G which is in the same measure class[7] as the Haar measure. It is possible to define a measure $\bar{\mu}$ on the Borel sets of $H_1 \backslash G / H_2$ by $\bar{\mu}(F) = \mu(s^{-1}(F))$.

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[7]Has the same set of null measure.

We shall call $\bar{\mu}$ an admissible measure in $H_1\backslash G/H_2$. The definition is well posed since any two of such measures have the same null measure sets.

General theory of induced representations can be found for instance in [Ma52, Fol16, BR86]. We recall the Mackey's subgroup theorem.

Theorem 4.8 (Mackey's subgroup theorem). *[Ma52]. Let H_1, H_2 be closed subgroups regularly related in G. Let π be a strongly continuous representation of H_1. For each $g \in G$ consider $H_g = H_2 \cap (g^{-1}H_1 g)$ and set*

$$\mathcal{V}_g = \mathrm{Ind}_{H_g}^{H_2}(\pi \circ \mathrm{ad}\, g).$$

Then $\mathcal{V}_g$ is determined to within equivalence by the double coset $[g]$ to which g belongs. If ν is an admissible measure on $H_1\backslash G/H_2$, then

$$(\mathrm{Ind}_{H_1}^{G}\pi)|_{H_2} \simeq \int_{H_1\backslash G/H_2}^{\oplus} \mathcal{V}_{[g]}\, d\nu([g]). \tag{4.1}$$

An immediate application of Theorem 4.8 to the restriction of $W_{r,n}$ to G_3 is the following lemma, which entered into the proof of Corollary 4.5 above.

Lemma 4.9. *The restriction of the $\tilde{G}_3$ representation $W_{r,n}$ to G_3 is $W_{r,m}$, where $m \equiv n \,(\mathrm{mod}\,2)$.*

Of central importance for our analysis is the following proposition.

Proposition 4.10. $U_h|_{\tilde{G}_3} \simeq \int_{\mathbb{R}^+}^{\oplus} W_{r,2h} dr$, *where $2h$ in the right-hand side has to be considered modulo 4.*

Proof. The h-helicity representation U_h is induced by the stabilizer $\overline{\mathrm{Stab}}_{q_1}$ of the point $q_1 = (1, 1, 0, 0)$. Again $\overline{\mathrm{Stab}}_{q_1}$ is isomorphic to $\mathbb{R}^{1+3} \rtimes \widetilde{\mathrm{E}}(2)$. In Theorem 4.8 we can consider $G = \widetilde{\mathcal{P}}_+^{\uparrow}$, $H_1 = \mathbb{R}^{1+3} \rtimes \widetilde{\mathrm{E}}(2)$ and we want to study the restriction of U_h to $H_2 = \tilde{G}_3$. We postpone the proof of the fact that H_1 and H_2 are regularly related.

Let us now compute H_g and $\mathcal{V}_g$ for several choices of g. First, for $g = \mathbf{1}$, $H_{g=\mathbf{1}} = \mathbb{R}^4 \rtimes \langle r_1(\pi) \rangle \subset \mathbb{R}^4 \rtimes \widetilde{\mathrm{E}}(2)$. Here we made use of the fact that $\tilde{G}_3^0 = \langle \lambda_3, r_3, r_1(\pi) \rangle$ and $r_1(\pi) \in \mathrm{Stab}_{q_1} = \widetilde{\mathrm{E}}(2)$. Hence

$$\mathcal{V}_{\mathbf{1}} = \mathrm{Ind}_{\mathbb{R}^4 \rtimes \langle r_1(\pi) \rangle}^{\tilde{G}_3}(\chi_{q_1} \cdot V_{2h}) = W_{1,2h},$$

by (3.12) and the comment below this formula.

Next, we consider $g = \lambda_1(\ln r) \in \widetilde{\mathcal{P}}_+^{\uparrow}$, where λ_1 is the lift of Λ_1, the boost in the x_1-direction, to $\mathrm{SL}(2, \mathbb{C})$, see Sect. 2.1. Setting $t \mapsto \Lambda_1(t) = \Lambda(\lambda_1(t))$ and noting that $q_r = \Lambda_1(\ln r)q_1 = r \cdot q_1$, the intersection $H_g := H_2 \cap (g^{-1}H_1 g)$ satisfies

$$H_g = H_{\mathbf{1}} \quad \text{since} \quad \tilde{G}_3 \cap \left(\lambda_1(\ln r)^{-1}\overline{\mathrm{Stab}}_{q_1}\lambda_1(\ln r)\right) = \tilde{G}_3 \cap \overline{\mathrm{Stab}}_{q_{r^{-1}}} = \mathbb{R}^4 \rtimes \langle r_1(\pi) \rangle.$$

Hence,

$$\begin{aligned}
\mathcal{V}_{\Lambda_1(\ln r)} &= \mathrm{Ind}_{\mathbb{R}^4 \rtimes \langle r_1(\pi) \rangle}^{\tilde{G}_3}\left((\chi_{q_1} \cdot V_{2h}) \circ \mathrm{ad}\, \lambda_1(\ln r)\right) \\
&= \mathrm{Ind}_{\mathbb{R}^4 \rtimes \langle r_1(\pi) \rangle}^{\tilde{G}_3}(\chi_{q_{r^{-1}}} \cdot V_{2h}) = W_{r^{-1},2h}
\end{aligned} \tag{4.2}$$

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because $\lambda_1 (\ln r)$ commutes with $r_1(\pi)$ (see comment on dual action in Sect. 3.4).

Finally, we consider $g = r_2(-\pi/2)$. We note for future reference that $\Lambda(g)^{-1}q_1 \in \sigma_0$ since $\Lambda(r_2(\pi/2))q_1 = R_2(\pi/2)q_1 = (1,0,0,1) =: q_0 \in \sigma_0$. We have

$$H_{r_2(-\pi/2)} = \tilde{G}_3 \cap r_2(\pi/2) \, \overline{\mathrm{Stab}}_{q_1} \, r_2(\pi/2)^{-1} = \tilde{G}_3 \cap \overline{\mathrm{Stab}}_{q_0} = \mathbb{R}^{1+3} \rtimes \langle r_3 \rangle,$$

$$\mathcal{V}_{r_2(-\pi/2)} = \mathrm{Ind}_{H_{r_2(-\pi/2)}}^{\tilde{G}_3} \left((\chi_{q_1} \cdot V_{2h}) \circ \mathrm{ad}\, r_2(\pi/2)^{-1} \right) = \mathrm{Ind}_{H_{r_2(-\pi/2)}}^{\tilde{G}_3} (\chi_{q_0} \cdot \chi_{2h})$$

and the joint spectrum of translations is supported in σ_0. Here χ_{2h} is the $2h$–character representation of $r_3(\,\cdot\,)$, $\chi_{2h}(r_3(\theta)) = e^{i\frac{\theta}{2}2h}$, $h \in \frac{\mathbb{Z}}{2}$.

Let us now we show that $[\lambda_1(t)]$ and $[r_2(-\pi/2)]$ cover all the equivalence classes in $H_1 \backslash G / H_2$. Let $g_1, g_2 \in \widetilde{\mathcal{P}}_+^{\uparrow}$ and assume that p_a and p_b are points on the same massless $\tilde{G}_3$-orbit[8] s.t. $\Lambda(g_1^{-1})q_1 = p_a$ and $\Lambda(g_2^{-1})q_1 = p_b$. That is, there exists $x \in \tilde{G}_3$ such that $p_b = \Lambda(x)p_a$. Then $g_1 x^{-1} g_2^{-1} = \mathrm{s} \in \overline{\mathrm{Stab}}_{q_1}$, thus $g_2 = \mathrm{s}^{-1}g_1 x^{-1}$ and g_2 belongs to $[g_1]$, the double coset of g_1 in $H_1 \backslash \widetilde{\mathcal{P}}_+^{\uparrow} / H_2$. In particular, for every $g \in G$ for which there exists $t_g \in \mathbb{R}$ such that $\Lambda(g^{-1})q_1$ and $\Lambda_1(-\ln t_g)q_1$ belong to the same orbit $\sigma_{t_g^{-1}}$, we have $g \in [\lambda_1(\ln t_g)]$. All the other $g \in G$ such that $\Lambda(g)^{-1}q_1 \in \sigma_0$ belong to the double coset $[r_2(-\pi/2)]$. This follows from the fact that $\Lambda(r_2(-\pi/2))^{-1}q_1 = q_0 \in \sigma_0$ which was mentioned above.

Now we will verify that the sets H_1 and H_2 are regularly related. For this purpose, we identify $H_1 \backslash G / H_2$ with $\mathbb{R}^+ = [0, +\infty)$ when $[\lambda_1(\ln r)] \in H_1 \backslash G / H_2$ is identified with $r^{-1} \in \mathbb{R}^+$ and $[r_2(-\pi/2)]$ with 0. We consider $\mathbb{R}^+$ with the relative topology inherited from $\mathbb{R}$. The quotient topology w.r.t. the map $s : G \to H_1 \backslash G / H_2$ defined by the previous identification guarantees that intervals of $\mathbb{R}^+$ are open sets. Indeed, let $\epsilon > 0$ and take an element g in the preimage $s^{-1}(I)$, $I := \mathbb{R}^+ \cap (a, b)$. Then the set $\mathcal{N}_g^{\epsilon} = \{g' \in G : \Lambda(g'^{-1})q_1 \in B_{\epsilon}(\Lambda(g^{-1})q_1\}$ is an open neighbourhood of g: as the inverse image of the open set $B_{\epsilon}(\Lambda(g^{-1})q_1)$ under the continuous mapping $g' \mapsto \Lambda(g'^{-1})q_1$, the set $\mathcal{N}_g^{\epsilon}$ is open and contains g. For sufficiently small ϵ it is contained in $s^{-1}(I)$, since if $\Lambda(g^{-1})q_1 \in \sigma_r$, then, for every $g' \in \mathcal{N}_g^{\epsilon}$, $\Lambda(g'^{-1})q_1 \in \sigma_{r'}$ with $r, r' \in I$ by continuity of the Poincaré action on Minkowski space. With the identification $H_1 \backslash G / H_2 \simeq \mathbb{R}^+$, it is easy to see that H_1 and H_2 are regularly related by using intervals $(q - \frac{1}{n}, q + \frac{1}{n}) \cap \mathbb{R}^+$, $n \in \mathbb{N}$, with rational center contained in $\mathbb{R}^+$. (The second part of Definition 4.7 is used here).

Let us now describe the equivalence class of an admissible measure (cf. Definition 4.7). Starting with a finite measure μ on G in the equivalence class of the Haar measure, we induce a measure on $\mathbb{R}^+ \simeq H_1 \backslash G / H_2$, which we prove to be in the measure class of the Lebesgue measure. Indeed, let W_r be the representation $W_{r^{-1}, 2h}$ and W_0 be $\mathcal{V}_{[r_2(-\pi/2)]}$, we get the formula:

$$U_h|_{\tilde{G}_3} \simeq \int_{\mathbb{R}^+}^{\oplus} d\mu(r) W_r. \tag{4.3}$$

Finite helicity representations U_h extend to the conformal group (cf. [Ma77]). In particular, by dilation covariance, we have that $U(\delta(t))U_h U(\delta(t))^* \simeq U_h$ hence

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[8]We have already one representative in each orbit, see above.

$U(\delta(t))U_h|_{\tilde{G}_3}U(\delta(t))^* \simeq U_h|_{\tilde{G}_3}$. Dilations change the unitary class of $W_{r,2h}$ dilating the radius of the representation[9], since

$$U(\delta(t))U_h|_{\tilde{G}_3}U(\delta(t))^* = U_h \circ \mathrm{ad}(\delta(t))|_{\tilde{G}_3} = \int_{\mathbb{R}^+}^{\oplus} d\mu(r)W_{e^{-t}r}. \tag{4.4}$$

Similarly as in Lemma 4.1 in [LMR16], this is a consequence of the following computation:

$$W_r \circ \mathrm{ad}(\delta(t)) = W_{r^{-1},2h} \circ \mathrm{ad}(\delta(t)) = \mathrm{Ind}_{\mathbb{R}^4 \rtimes \langle r_1(\pi) \rangle}^{\tilde{G}_3} \left((\chi_{q_1} \cdot V_{2h}) \circ \mathrm{ad}\,\lambda_1\,(\ln r) \circ \mathrm{ad}(\delta(t)) \right)$$

$$= \mathrm{Ind}_{\mathbb{R}^4 \rtimes \langle r_1(\pi) \rangle}^{\tilde{G}_3}(\chi_{q_{e^t r^{-1}}} \cdot V_{2h}) = W_{e^t r^{-1},2h} = W_{e^{-t}r}, \tag{4.5}$$

where $q_{e^t r^{-1}} := (e^t r^{-1}, e^t r^{-1}, 0, 0)$ and in the second step we used Lemma 4.1 of [LMR16]. An analogous computation can be done for W_0 in order to show that $W_0 \circ \mathrm{ad}(\delta(t)) \simeq W_0$: since $\mathrm{ad}(\lambda_3)$ does not change the unitary equivalence class of the $\tilde{G}_3$-representations

$$\mathcal{V}_{[r_2(-\pi/2)]} \circ \mathrm{ad}(\lambda_3(t)) \circ \mathrm{ad}(\delta(t)) = \mathrm{Ind}_{H_{r_2(\pi/2)}}^{\tilde{G}_3}(\chi_{q_0} \cdot \chi_{2h}) \circ \mathrm{ad}(\lambda_3(t)) \circ \mathrm{ad}(\delta(t)) =$$

$$= \mathrm{Ind}_{H_{r_2(\pi/2)}}^{\tilde{G}_3}(\chi_{q_0} \cdot \chi_{2h}) = \mathcal{V}_{[r_2(-\pi/2)]},$$

where we used $(\chi_{q_0} \cdot \chi_{2h}) \circ \mathrm{ad}(\lambda_3(t)) \circ \mathrm{ad}(\delta(t)) = (\chi_{\lambda_3(-t)\delta(t)q_0} \cdot \chi_{2h}) = \chi_{q_0} \cdot \chi_{2h}$ referring again to Lemma 4.1 of [LMR16]. Therefore,

$$\int_{\mathbb{R}^+}^{\oplus} d\mu(r)W_r \simeq \int_{\mathbb{R}^+}^{\oplus} d\mu(r)W_{e^t r} = \int_{\mathbb{R}^+}^{\oplus} d\mu_t(r)W_r,$$

where $\mu_t(r) := \mu(e^{-t}r)$. We show that μ is equivalent to the Lebesgue measure: assume by contradiction that there exists a set $E \subset \mathbb{R}^+ \backslash \{0\}$ such that $\mu(E) > 0$ but $\mu_t(E) = 0$ and consider the multiplication operator by the projection $P_E := \int_E^{\oplus} d\mu(r) \in U_h(\tilde{G}_3)'$. Then we have that the subrepresentation $P_E U_h|_{\tilde{G}_3} P_E$ is not contained in $U(\delta(t))U_h|_{\tilde{G}_3}U(\delta(t))^*$ (representations of radius $r \in E$ have measure zero in the latter representation). In particular for every $t \in \mathbb{R}$, μ_t is equivalent to μ, hence to the Lebesgue measure on $\mathbb{R}^+ \backslash \{0\}$ up to a possible singular measure in 0 (see Proposition 11 of [Bo04]). Since σ_0 has null measure in the joint spectrum of translation in U_h, by comparing the translation spectrum in left and right side of (4.3), $\{r = 0\}$ has null μ−measure. Now the statement of the theorem is obtained by a change of variables $r \mapsto \frac{1}{r}$. $\qquad \square$

By Propositions 4.6, 4.10 and 4.4 (ii) we conclude the modularity condition for finite helicity representations:

Corollary 4.11. *For every $h \in \frac{\mathbb{Z}}{2}$, U_h and its multiples satisfy* (MC).

Proposition 4.12. *If h is an integer, namely U_h is bosonic, then any finite or infinite multiple of $U_h \oplus U_{-h}$ satisfies* (MC).

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[9]It is easy to see that the dual action of $\delta(t)$ on a character χ_p is given by $\delta(t)p = e^t p$.

Proof. By Proposition 4.10 U_h and U_{-h} have unitarily equivalent restrictions to $\tilde{G}_3$. Indeed, since h is supposed to be integer, $2h$ is equal to 0 or 2 modulo 4. Clearly $0 \equiv -0 \pmod 4$ and $2 \equiv -2 \pmod 4$. By Proposition 4.6 and Proposition 4.4 (ii),(iii), the direct sum $U_h \oplus U_{-h}$ satisfies (MC). Any multiple of U satisfies (MC) again by Proposition 4.4 *(ii)*. $\qquad\square$

The main result of this section is a corollary of Theorem 4.2:

Theorem 4.13. *Every net of real subspaces H undergoing the action of a finite or infinite multiple of $U = U_h \oplus U_{-h}$, where $h \in \mathbb{Z}$, satisfies the B-W and the duality properties.*

A final remark on finite helicity one particle nets is the following. Massless non-zero finite helicity representations of the Poincaré group have to be properly coupled in order to act consistently on a net of standard subspaces on spacelike cones[10]. Indeed, by Theorem 4.2, U_h satisfies the modularity condition (MC) and any net of standard subspaces it acts covariantly on satisfies the B-W property. Following [GL95], when the B-W property holds then by spacelike cone localization property one deduces that U_h is covariant under the action of the wedge modular conjugations, namely U_h extends to an (anti-)unitary representation of the group $\widetilde{\mathcal{P}}_+ = \langle \widetilde{\mathcal{P}}_+^\uparrow, \Theta \rangle$, where Θ is the space and time reflection. The extension is unique up to unitary equivalence (Proposition 2.3 of [LMPR] or [NO17] for an abstract discussion). This is not possible for the irreducible finite helicity representations as they are not induced by a selfconjugate representation of the little group (cf. for instance [Va85]). In particular any anti-unitary operator implementing the PT symmetry (no charge C considered in this one particle setting) takes U_h into U_{-h}.

5 Bisognano-Wichmann property and asymptotic completeness

In this section we apply Theorem 4.13 to a concrete one-particle net of standard subspaces in the subspace $\mathfrak{H}^{(1)}$ from Definition 2.3 and then verify the Bisognano-Wichmann property on $\mathfrak{H}^{\text{out}}$ using scattering theory.

Let $\mathcal{A}_{\text{sa}}(W) \subset \mathcal{A}(W)$ be the subspace of self-adjoint operators. It is well known and easy to check that if $(\mathcal{A}, U, \Omega)$ is a local net of von Neumann algebras in the sense of Definition 2.1, then

$$H(W) := \overline{\{\, A\Omega \,:\, A \in \mathcal{A}_{\text{sa}}(W) \,\}}, \tag{5.1}$$

is a net of standard subspaces on wedges w.r.t. U, i.e., it satisfies properties 1.-5. of Sect. 3.3. Motivated by formula (2.14), we define for any $W \in \mathcal{W}$ the following real subspace of $\mathfrak{H}^{(1)}$

$$H^{(1)}(W) := P^{(1)} H(W). \tag{5.2}$$

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[10]A spacelike cone is a set of the form $C = a + \cup_{\lambda > 0} \lambda O$ where $a \in \mathbb{R}^{1+3}$ is the apex and O is a double cone which contains only spacelike points and its closure does not contain the origin. It is the intersection of finitely many wedge regions.

Furthermore, we recall that by the Borchers theorem $\Delta_{\mathcal{A}(W)}, J_{\mathcal{A}(W)}, S_{\mathcal{A}(W)}$ commute with the mass operator and therefore can be restricted to the domain

$$D := \{ P^{(1)} A\Omega : A \in \mathcal{A}(W) \}, \tag{5.3}$$

which is dense in $\mathfrak{H}^{(1)}$. (This is proven analogously as Corollary 3.10). The following result holds:

Proposition 5.1. *Let $(\mathcal{A}, U, \Omega)$ be a local net of von Neumann algebras describing massless particles.*

(i) *The map $H^{(1)}(W) = P^{(1)} H(W)$, defined in (5.1)-(5.2) above, gives a one particle net of standard subspaces on wedges w.r.t. $U^{(1)} := P^{(1)} U P^{(1)}$, in the sense of properties 1.-5. of Sect. 3.3.*

(ii) *If the theory $(\mathcal{A}, U, \Omega)$ contains massless Wigner particles with helicity zero or with helicities $(h, -h)$, $h \in \mathbb{N}$, then the one-particle net $W \mapsto H^{(1)}(W)$ satisfies the B-W and duality properties.*

(iii) $J_{H^{(1)}(W)} = J_{\mathcal{A}(W)} P^{(1)}$ *and* $\Delta^{it}_{H^{(1)}(W)} = \Delta^{it}_{\mathcal{A}(W)} P^{(1)}$ *for $t \in \mathbb{R}$.*

Remark 5.2. *The locality property for the net $W \mapsto H^{(1)}(W)$ can be extracted from [Bu77], where*

$$\langle \Omega, A_1 P^{(1)} A_2 \Omega \rangle = \langle \Omega, A_2 P^{(1)} A_1 \Omega \rangle \tag{5.4}$$

was obtained for A_1, A_2 localized in spacelike separated double cones by the JLD technique. In our context the same property follows from the Borchers theorem and Lemma 3.9.

Proof. (i) By Lemmas 3.7, 3.8 and 3.9 we have that $H(W) = P^{(1)} H(W) \oplus (1 - P^{(1)}) H(W)$ and $W \mapsto P^{(1)} H(W)$ defines a local net of standard subspaces on $\mathfrak{H}^{(1)} := P^{(1)} \mathfrak{H}$. It transforms under the massless Poincaré representation $P^{(1)} U P^{(1)}$ and satisfies the assumptions 1.-5 in Sect. 3.3.

(ii) For helicities as in the statement of the proposition we obtain the B-W property for the one-particle net from Theorem 4.13.

(iii) Let $\xi_i \in H^{(1)}(W)$, $i = 1, 2$, and $P^{(1)} A_{i,n} \Omega$, $n \in \mathbb{N}$, be the corresponding approximating sequences with $A_{i,n}^* = A_{i,n}$. Then

$$\begin{aligned} S_{H^{(1)}(W)}(\xi_1 + i\xi_2) = \xi_1 - i\xi_2 &= \lim_{n\to\infty} P^{(1)}(A_{1,n}\Omega - iA_{2,n})\Omega \\ &= \lim_{n\to\infty} P^{(1)} S_{\mathcal{A}(W)}(A_{1,n}\Omega + iA_{2,n})\Omega \\ &= \lim_{n\to\infty} S_{\mathcal{A}(W)} P^{(1)}(A_{1,n}\Omega + iA_{2,n})\Omega. \end{aligned} \tag{5.5}$$

Hence $H^{(1)}(W) + iH^{(1)}(W)$ belongs to the domain of the closure of $S_{\mathcal{A}(W)} P^{(1)}$ and the latter operator coincides with $S_{H(W)}$ on $H^{(1)}(W) + iH^{(1)}(W)$. By the uniqueness of the polar decomposition, we have $J_{H^{(1)}(W)} = J_{\mathcal{A}(W)} P^{(1)}$ and $\Delta^{it}_{H^{(1)}(W)} = \Delta^{it}_{\mathcal{A}(W)} P^{(1)}$. $\square$

After this preparation we give a massless version of Lemma 6 and Proposition 7 from [Mu01] and thereby conclude the proof of the Bisognano-Wichmann property for asymptotically complete massless theories, as stated in Theorem 2.5.

Lemma 5.3. *In each n-particle subspace $\mathfrak{H}^{(n)}$, $n \geq 2$, there is a total set of scattering states $\Psi^{\mathrm{out}} := A_n^{\mathrm{out}}\{f_n\}^+ \ldots A_1^{\mathrm{out}}\{f_1\}^+\Omega$ with the localisation regions $\mathcal{O}_i$ of A_i chosen s.t. $\mathcal{O}_n \subset W_1$. Furthermore, $(\boldsymbol{v}_i - \boldsymbol{v}_n)_1 < 0$ for $\boldsymbol{v}_i \in \operatorname{supp} f_i$ and $\boldsymbol{v}_n \in \operatorname{supp} f_n$ for $i \neq n$.*

Proof. We consider an arbitrary scattering state $\Psi^{\mathrm{out}} = (\Phi_n \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \Phi_1)$ constructed using $A_1, \ldots, A_n$ localized in some arbitrary double cones $\tilde{\mathcal{O}}_1, \ldots, \tilde{\mathcal{O}}_n$ and arbitrary smooth functions $f_1, \ldots f_n$ on S^2. By cutting the sphere of velocities into small slices with planes orthogonal to the 1-st axis, we can approximate Ψ^{out} with linear combinations of scattering states Ψ_1^{out} such that the projections of $\operatorname{supp} f_i$ on the 1-st axis are disjoint. (Cf. Proposition 3.2, formula (2.14) and the absolute continuity of the momentum spectral measure [BF82]). Let f_{i_0} be such function that

$$(\boldsymbol{v}_i - \boldsymbol{v}_{i_0})_1 < 0 \tag{5.6}$$

for all $\boldsymbol{v}_{i_0} \in \operatorname{supp} f_{i_0}$ and $\boldsymbol{v}_i \in \operatorname{supp} f_i$, $i \neq i_0$. Up to numbering, these scattering states satisfy the condition from the lemma concerning velocities, but possibly not the condition concerning localisation regions. Therefore, using again Proposition 3.2, formula (2.14) and the Reeh-Schlieder property for wedges, we approximate each Ψ_1^{out} by linear combinations of vectors Ψ_2^{out} constructed using A_i' localised in double cones $\mathcal{O}_i$ s.t. $\mathcal{O}_{i_0} \in W_1$. Due to the canonical commutation relations of asymptotic fields (cf. formula (3.2) above) each such vector $\Psi_2^{\mathrm{out}} = (\Phi_n' \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \Phi_1')$ coincides with $(\Phi_{i_0}' \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \Phi_n' \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \Phi_1')$. By changing the numbering, we obtain the claim. $\square$

Proposition 5.4. *If the unitary groups $\mathbb{R} \ni s \mapsto \Delta_{\mathcal{A}(W_1)}^{is}$ and $\mathbb{R} \ni s \mapsto U(\lambda_{W_1}(-2\pi s))$ coincide on $\mathfrak{H}^{(1)}$, they also coincide on the subspace $\mathfrak{H}^{\mathrm{out}}$ of scattering states.*

Proof. Let $V_s := \Delta_{\mathcal{A}(W_1)}^{is} U(\Lambda_{W_1}(2\pi s))$. By induction over the particle number n, we show that V_s is the unity on each $\mathfrak{H}^{(n)}$. Let $\Psi^{\mathrm{out}} = (\Phi_n \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \Phi_1)$ be a scattering state as in Lemma 5.3 with $\Phi_i = A_i^{\mathrm{out}}\{f_i\}\Omega$. For $n = 0$ we have $\Phi^{\mathrm{out}} = \Omega$ and the statement follows from the invariance of the vacuum under $U(\cdot)$ and $s \mapsto \Delta_{\mathcal{A}(W_1)}^{is}$. For $n = 1$ the statement holds by Proposition 5.1 (ii). Now let $n \geq 2$ be arbitrary and suppose that the statement holds for $n' < n$. Making use of Proposition 3.1 (d), we write

$$\begin{aligned}
\Psi^{\mathrm{out}} &= A_n^{\mathrm{out}}\{f_n\}^+ \ldots A_1^{\mathrm{out}}\{f_1\}^+\Omega \\
&= (A_n^{\mathrm{out}}\{f_n\}^+ + A_n^{\mathrm{out}}\{f_n\}^-) \ldots (A_1^{\mathrm{out}}\{f_1\}^+ + A_1^{\mathrm{out}}\{f_1\}^-)\Omega + \check{\Psi}^{\mathrm{out}} \\
&= A_n^{\mathrm{out}}\{f_n\} \ldots A_1^{\mathrm{out}}\{f_1\}\Omega + \check{\Psi}^{\mathrm{out}},
\end{aligned} \tag{5.7}$$

where, by the canonical commutation relations for the asymptotic creation/annihilation operators (cf. formula (3.2)), the compensating vector $\check{\Psi}^{\mathrm{out}}$ has components only in $\mathfrak{H}^{(\ell)}$ for $\ell < n$. Thus we have, by the induction hypothesis, $V_s\check{\Psi}^{\mathrm{out}} = \check{\Psi}^{\mathrm{out}}$, and it suffices to consider $\hat{\Psi}^{\mathrm{out}} := A_n^{\mathrm{out}}\{f_n\} \ldots A_1^{\mathrm{out}}\{f_1\}\Omega$. We can write

$$\begin{aligned}
V(s)\hat{\Psi}^{\mathrm{out}} &= V_s A_n^{\mathrm{out}}\{f_n\} \ldots A_1^{\mathrm{out}}\{f_1\}\Omega \\
&= V_s A_n^{\mathrm{out}}\{f_n\} V_s^{-1} V_s A_{n-1}^{\mathrm{out}}\{f_{n-1}\} \ldots A_1^{\mathrm{out}}\{f_1\}\Omega \\
&= V_s A_n^{\mathrm{out}}\{f_n\} V_s^{-1} A_{n-1}^{\mathrm{out}}\{f_{n-1}\} \ldots A_1^{\mathrm{out}}\{f_1\}\Omega, \tag{5.8}
\end{aligned}$$

where in the last step we used the induction hypothesis. (We also used that the state $A^{\text{out}}_{n-1}\{f_{n-1}\}\ldots A^{\text{out}}_1\{f_1\}\Omega$ is in D_{P^0}, cf. Proposition 3.1, and the fact that due to the Borchers theorem, V_s leaves D_{P^0} invariant. Consequently, the limit $A^{\text{out}}_n\{f_n\}$ still exists in the last line above). Using again that V_s commutes with translations, we have $V_s A^{\text{out}}_n\{f_n\}V_s^{-1} = (A'_n)^{\text{out}}\{f_n\}$, where $A'_n := V_s A_n V_s^{-1}$ is localized in W_1. Thus we can write

$$V(s)\hat{\Psi}^{\text{out}} = (A'_n)^{\text{out}}\{f_n\}A^{\text{out}}_{n-1}\{f_{n-1}\}\ldots A^{\text{out}}_1\{f_1\}\Omega$$

$$= \sum_{i=n-1}^{1} A^{\text{out}}_{n-1}\{f_{n-1}\}\ldots[(A'_n)^{\text{out}}\{f_n\}, A^{\text{out}}_i\{f_i\}]\ldots A^{\text{out}}_1\{f_1\}\Omega \tag{5.9}$$

$$+ A^{\text{out}}_{n-1}\{f_n\}\ldots A^{\text{out}}_1\{f_1\}(A'_n)^{\text{out}}\{f_n\}\Omega. \tag{5.10}$$

Concerning (5.10), we use that $(A'_n)^{\text{out}}\{f_n\}\Omega = V_s A^{\text{out}}_n\{f_n\}V_s^{-1}\Omega = A^{\text{out}}_n\{f_n\}\Omega$, since V_s preserves both the vacuum and single-particle subspace. This vector coincides with $\hat{\Psi}^{\text{out}}$ provided that we can show

$$[A^{\text{out}}_n\{f_n\}, A^{\text{out}}_i\{f_i\}] = 0 \tag{5.11}$$

for all $i = 1, 2, \ldots, n-1$. This follows from formulas (2.14), (2.15) and the disjointness of supports of f_n, f_i (cf. Lemma 5.3). Thus to conclude the proof of the proposition we have to show that the terms in (5.9) are zero. Since A'_n is only wedge-local, we cannot use property (2.15) and we need to proceed via a direct computation: For unit vectors $\Psi_1, \Psi_2 \in D_{P^0}$ we write

$$|\langle\Psi_1[(A'_n)^{\text{out}}\{f_n\}, A^{\text{out}}_i\{f_i\}]\Psi_2\rangle|$$

$$\leq \overline{\lim}_{t\to\infty} \int dt'_n dt'_i d\omega(\boldsymbol{n}_n)d\omega(\boldsymbol{n}_i)\, h_t(t'_n)h_t(t'_i)|f_n(\boldsymbol{n}_n)||f_i(\boldsymbol{n}_i)4t'_n t'_i \times$$

$$\times \|[\partial_0 A'_n(t'_n, t'_n\boldsymbol{n}_n), \partial_0 A_i(t'_i, t'_i\boldsymbol{n}_i)]\|$$

$$\leq \overline{\lim}_{t\to\infty} \int dt'_n dt'_i d\omega(\boldsymbol{n}_n)d\omega(\boldsymbol{n}_i)\, h_t(t'_n)h_t(t'_i)|f_n(\boldsymbol{n}_n)||f_i(\boldsymbol{n}_i)4t'_n t'_i \times$$

$$\times \|[\partial_0 A'_n, \partial_0 A_i(t'_i - t'_n, t'_i\boldsymbol{n}_i - t'_n\boldsymbol{n}_n)]\|$$

$$\leq \overline{\lim}_{t\to\infty} \int dt''_n dt''_i d\omega(\boldsymbol{n}_n)d\omega(\boldsymbol{n}_i)\, t^{2\varepsilon}h(t''_n)h(t''_i)|f_n(\boldsymbol{n}_n)||f_i(\boldsymbol{n}_i)|4(t + t^\varepsilon t''_i)(t + t^\varepsilon t''_n) \times$$

$$\times \|[\partial_0 A'_n, \partial_0 A_i(t^\varepsilon(t''_i - t''_n), (t + t^\varepsilon t''_i)\boldsymbol{n}_i - (t + t^\varepsilon t''_n)\boldsymbol{n}_n)]\|. \tag{5.12}$$

Making use of Lemma 5.3, we conclude that the last expression is zero for sufficiently large t. Indeed, $t(\boldsymbol{n}_i - \boldsymbol{n}_n)_1 < 0$, $0 < \varepsilon < 1$ and t''_i, t''_n are restricted to unit balls around zero. Hence $\partial_0 A_i(t^\varepsilon(t''_i - t''_n), (t + t^\varepsilon t''_i)\boldsymbol{n}_i - (t + t^\varepsilon t''_n)\boldsymbol{n}_n)$ is eventually localized in the left wedge. $\qquad\square$

6 Conclusion and outlook

In this paper we proved the Bisognano-Wichmann property for asymptotically complete theories of massless particles with integer helicities. The argument starts from

verifying this property at the single particle level. For this purpose, the single-particle subspace $\mathfrak{H}^{(1)}$ is equipped with the structure of a local net of standard subspaces. This net is covariant w.r.t. a representation $U^{(1)} := U|_{\mathfrak{H}^{(1)}}$ of the Poincaré group, which is a direct sum of two representations of opposite integer helicities, i.e., $U^{(1)} = U_h \oplus U_{-h}$ (or a multiple thereof). Then we verified the modularity condition for the B-W property $U^{(1)}(r_W) \in U^{(1)}(G_W)''$, where G_W is the subgroup of Poincaré transformations preserving a wedge W and r_W maps W to the opposite wedge. This technically demanding step was accomplished by showing that U_h and U_{-h} have the same restriction to the group $\tilde{G}_W$ generated by G_W and r_W. Hence $U^{(1)}|_{\tilde{G}_W} = U_h|_{\tilde{G}_W} \otimes 1$ and the modularity condition could be concluded from earlier results [Mo18]. Given the B-W property at the single-particle level, the B-W property of the full theory was verified using scattering theory and asymptotic completeness. In this part we adapted the arguments of Mund [Mu01] to the massless case.

A natural question for future research is a generalization of our arguments to particles with half-integer helicities. The obstruction comes from the fact that in this case U_h and U_{-h} are unitarily equivalent when restricted to G_3 but have disjoint restrictions to $\tilde{G}_3$ and one cannot apply Proposition 4.4 (iii). Another future research direction is to relax the assumption of asymptotic completeness. We remark that in the vacuum sector of QED asymptotic completeness of photons can be assumed only below a certain energy threshold, excluding the electron-positron pair production. It is an interesting question how to prove the B-W property in this physically relevant situation. In this context we remark that our results give the B-W property of the net of asymptotic photon fields of QED, defined at the end of Sect. 2.2. This net plays an important role in the study of infrared problems (see e.g. [Bu77, BD84, AD17]) and we hope that our results will also find applications there.

A   Direct integral representations

We suggest [Tak02, Dix81, MT18] as further references for basic definitions.

Given a field of Hilbert spaces $\gamma \mapsto \mathcal{H}(\gamma)$ on a standard measure space (Γ, μ), the direct integral Hilbert space $\int_\Gamma^\oplus \mathcal{H}(\gamma)d\mu(\gamma)$ is defined if the field is μ-measurable. This definition requires and depends on the choice of a linear γ-pointwise dense subspace $\mathcal{S}$ of the topological product $\Pi_{\gamma \in \Gamma}\mathcal{H}(\gamma)$ which selects a family of μ-measurable vector fields. (cf. [Dix81, Part II, Sect. II.1.3, Definition 1]). Note that given a sequence of measurable vector field ξ_n μ-a.e. pointwise converging to ξ, namely $\|(\xi_n)(\gamma) - (\xi)(\gamma)\|_\gamma \to 0$ for μ-a.e. $\gamma \in \Gamma$, we obtain that ξ is a μ-measurable vector field. We also recall that a vector field of bounded operators $\gamma \mapsto T(\gamma) \in \mathcal{B}(\mathcal{H}(\gamma))$ is μ-measurable if for any μ-measurable field $\gamma \mapsto \xi(\gamma) \in \mathcal{H}(\gamma)$ we have that $\gamma \mapsto T(\gamma)\xi(\gamma) \in \mathcal{H}(\gamma)$ is μ-measurable. In this case we define

$$T\xi := \int_\Gamma^\oplus T(\gamma)\xi(\gamma)d\mu(\gamma) \in \int_\Gamma^\oplus \mathcal{H}(\gamma)d\mu(\gamma).$$

We write this operator as $T = \int_\Gamma^\oplus T(\gamma)d\mu(\gamma)$ and T is called the direct integral of $\gamma \mapsto T(\gamma)$. Furthermore, we have that $\gamma \mapsto \|T(\gamma)\|_\gamma$ is measurable and $\|T\| = \sup_{\gamma \in \Gamma} \|T(\gamma)\|_\gamma$. The operators of this form are said to be *decomposable*. If $T(\gamma)$ is a

scalar for any $\gamma \in \Gamma$, then T is said to be a *diagonal* operator. The algebra generated by the diagonal operators is called the *diagonal* algebra. Note that any operator $T \in \mathcal{B}(\mathcal{H})$ is decomposable iff T commutes with the diagonal algebra (cf. [Tak02] IV.8, Corollary 8.16). A field of von Neumann algebras $\gamma \mapsto \mathcal{M}(\gamma) \subset \mathcal{B}(\mathcal{H}(\gamma))$ is said to be measurable if there exists a countable family $\{x_n(\gamma)\}_{n \in \mathbb{N}}$ of measurable fields of operators s.t. $\mathcal{M}(\gamma)$ is generated by $x_n(\gamma)$ for (a.e.) $\gamma \in \Gamma$. Then it is possible to define $\mathcal{M} = \int_\Gamma^\oplus \mathcal{M}(\gamma) d\mu(\gamma)$. Given a C^*-algebra $\mathcal{A}$, a field of continuous representations $\gamma \mapsto \pi(\gamma)$ is said to be measurable if for any $A \in \mathcal{A}$, the operator field $\gamma \mapsto \pi(\gamma)(A)$ is measurable. Then one can define the representation $\int_\Gamma^\oplus \pi(\gamma) d\mu(\gamma)$ acting on $\int_\Gamma^\oplus \mathcal{H}(\gamma) d\mu(\gamma)$.

B Proof of Proposition 4.6

For fundamental concepts on direct integral of representation see Appendix A.

The function of the translation generators $P_1^2 + P_2^2$ is a Casimir operator for G_3 and decomposes according to μ, i.e., $P_1^2 + P_2^2 = \int_{\mathbb{R}^+}^\oplus r^2 \cdot 1 d\mu(r)$ and $P_1^2 + P_2^2$ is affiliated to $U(G_3)''$. By definition, bounded functions of $r = \sqrt{P_1^2 + P_2^2}$ generate the diagonal algebra $\mathcal{D}$. Thus $\mathcal{D}$ is contained in the center of $U(G_3)''$, hence any operator in $U(G_3)'$ is decomposable since it commutes with $\mathcal{D}$.

We now pick $T \in U(G_3)'$ then T is a decomposable operator, namely,

$$T = \int_{\mathbb{R}^+}^\oplus T(r) d\mu(r). \tag{B.1}$$

Assume that there exists a positive measure set I s.t. $T(r)$ is not in $U_r(G_3)'$ for $r \in I$. Let χ be a characteristic function of I. Then $\int_{\mathbb{R}^+}^\oplus T(r)\chi(r) d\mu(r)$ is not in the commutant of $U(G_3)$, which is a contradiction. We conclude that

$$U(G_3)' = \int_{\mathbb{R}^+}^\oplus U_r(G_3)' d\mu(r) \tag{B.2}$$

and thus

$$U(G_3)'' = \int_{\mathbb{R}^+}^\oplus U_r(G_3)'' d\mu(r) \tag{B.3}$$

by [Tak02, Theorem 8.18]. Now we recall that U_r satisfies (MC) by Corollary 4.5. Since, by assumption, $U(r_1(\pi)) = \int_{\mathbb{R}^+}^\oplus U_r(r_1(\pi)) d\mu(r)$, we obtain from (B.3) that U satisfies (MC).

C Regularity of the actions on $\mathbb{R}^{1+3}$

Definition C.1. *Let G be a locally compact, σ-compact, group and N be a normal abelian subgroup, then the (dual-)action of G on $\hat{N}$ is regular if*

R1. the orbit space is countably separated, namely there exists a countable family $\{E_n\}_{n \in \mathbb{N}}$ of G-invariant Borel sets in $\hat{N}$ s.t. each orbit in $\hat{N}$ is the intersection of all E_n that contain it,

R2. each orbit is relatively open in its closure.

Here we check that the action of G_3^0 and $\tilde{G}_3^0$ on $\mathbb{R}^{1+3}$ is regular according to the previous definition.

R1. Let $\mathbf{o}$ be a G_3^0 or a $\tilde{G}_3^0$ orbit on $\mathbb{R}^{1+3}$. Then $\mathbf{o}$ can be obtained by intersection of the subsets of the following *countable* family of Borel subsets containing $\mathbf{o}$. For $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, $c, d \in \mathbb{Q}^{\geq 0}$, consider the following sets:

- $A_{a_1, b_1} = \{p : a_1 \leq p^2 \leq b_1\}$ and $A_{a_1, b_1}^{\pm} = A_{a_1, b_1} \cap \{p : \pm p_0 > 0\}$ if $a_1, b_1 \geq 0$,

- $E_{c,d} = \{p : c \leq p_1^2 + p_2^2 \leq d\}$,

- $F_{a_2, b_2} = \{p : a_2 \leq p_0^2 - p_3^2 \leq b_2\}$,

- $F_{a_2, b_2}^{\pm} = \{p : a_2 \leq p_0^2 - p_3^2 \leq b_2, \pm p_3 > 0\}$ if $a_2, b_2 < 0$,

- $K^{\pm, \pm} = \{p = (p_0, 0, 0, p_3) : p_0 = \pm p_3, \pm p_0 > 0\}$,

- $\tilde{K}^{\pm} = \{p = (p_0, 0, 0, p_3) : p_0 = \pm |p_3|, \pm p_0 > 0\}$,

where in the case of $K^{\pm, \pm}$ the two signs are uncorrelated. We shall denote with $\mathcal{A}, \mathcal{E}, \mathcal{F}, \mathcal{F}^{\pm}, \mathcal{K}^{\pm\pm}, \tilde{\mathcal{K}}^{\pm}$ the countable families of the above sets with the corresponding letters. We also define the sets:

- $O = \{p : p = 0\}$,

- $Z_{a_1, b_1, \pm, \pm} = \{p : a_1 \leq p^2 \leq b_1 < 0, p_0^2 - p_3^2 = 0, \pm p_3 > 0, \pm p_0 > 0\}$,

- $\tilde{Z}_{a_1, b_1, \pm} = \{p : a_1 \leq p^2 \leq b_1 < 0, p_0^2 - p_3^2 = 0, \pm p_0 > 0\}$,

- $X_{a_1, b_1} = \{p : a_1 \leq p^2 \leq b_1 < 0, p_0 = p_3 = 0\}$.

In the following we shall say that a family of sets *selects* an orbit $\mathbf{o}$ if the latter is the intersection of all the set of the family containing $\mathbf{o}$. The set selecting an orbit of a group will be invariant under the group action. All the families we will consider will be countable as well as their union.

Firstly, any orbit of G_3^0 or $\tilde{G}_3^0$ is contained in a Lorentz orbit in $\mathbb{R}^{1+3}$. The family in $\mathcal{A}$ selects the Lorentz orbits. The orbit in the origin is selected by O. Now G_3^0 and $\tilde{G}_3^0$ share the same massive orbits, contained in $p^2 = m^2$, $m > 0$, that can be selected by considering $\mathcal{A}$ and $\mathcal{E}$ families. Now consider a massless orbit in the forward lightcone. If for every $p \in \mathbf{o}$, $p_1^2 + p_2^2 > 0$ then it is both a G_3^0 and $\tilde{G}_3^0$ orbit (cf. Remark 3.13) and can be selected by the families $\mathcal{A}$ and $\mathcal{E}$. If $p_1^2 + p_2^2 = 0$ then the two G_3^0 orbits $\{p : 0 < p_0 = p_3\}$ and $\{p : 0 < p_0 = -p_3\}$ are selected by $K^{\pm, +}$. If $p_1^2 + p_2^2 = 0$ then the $\tilde{G}_3^0$ orbit is selected by $\tilde{K}^+$. We argue analogously for the backward lightcone, referring to sets $K^{\pm, -}, \tilde{K}^-$. Now consider imaginary mass orbits, contained in $p^2 = -m^2$ and assume that $p_1^2 + p_2^2 = r^2$. We have three cases:

- $r^2 < -m^2$. In this case we have two branches of the hyperboloid $p_0^2 - p_3^2 = m^2 + r^2 < 0$ that become two G_3^0 orbits and a unique $\tilde{G}_3^0$ orbit. The G_3^0 and $\tilde{G}_3^0$ orbits are selected by $\mathcal{A}, \mathcal{E}, \mathcal{F}^{\pm}$ and $\mathcal{A}, \mathcal{E}, \mathcal{F}$, respectively.

- $r^2 = -m^2$. G_3^0 orbits are selected by $Z_{a_1,a_2,\pm,\pm}$ or X_{a_1,a_2}. $\tilde{G}_3^0$ orbits are selected by $\tilde{Z}_{a_1,a_2,\pm}$.

- $r^2 > -m^2$. G_3 and $\tilde{G}_3$ share the same orbits selected by $\mathcal{A}, \mathcal{E}, \mathcal{F} \cap U^\pm$, where $U^\pm = \{p : \pm p_0 \geq 0\}$.

R2. trivially holds.

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